Probabilistic Robotics

Bayes Filter Implementations

Gaussian filters

Bayes Filter Reminder

z = observation
u = action
x = state

$$\boxed{Bel(x_t)} = P(x_t \mid u_1, z_1 \ldots, u_t, z_t)$$

Bayes
$$= \eta\ P(z_t \mid x_t, u_1, z_1, \ldots, u_t)\ P(x_t \mid u_1, z_1, \ldots, u_t)$$

Markov
$$= \eta\ P(z_t \mid x_t)\ P(x_t \mid u_1, z_1, \ldots, u_t)$$

Total prob.
$$= \eta\ P(z_t \mid x_t) \int P(x_t \mid u_1, z_1, \ldots, u_t, x_{t-1})\ P(x_{t-1} \mid u_1, z_1, \ldots, u_t)\ dx_{t-1}$$

Markov
$$= \eta\ P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1})\ P(x_{t-1} \mid u_1, z_1, \ldots, u_t)\ dx_{t-1}$$

Markov
$$= \eta P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1})\, P(x_{t-1} \mid u_1, z_1, \ldots, z_{t-1})\ dx_{t-1}$$

$$\boxed{= \eta\ P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1})\ Bel(x_{t-1})\ dx_{t-1}}$$

$$Bel(x_t) = \eta \ P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \ Bel(x_{t-1}) \ dx_{t-1}$$

1. Algorithm **Bayes_filter**(*Bel*(x_{t-1}),u_t,z_t):
2. For all x_t do
3. $\quad \overline{bel}(x_t) = \int p(x_t \mid u_t, \underline{x_{t-1}}) \ bel(x_{t-1}) \ dx_{t-1}$
4. $\quad bel(x_t) = \eta \ p(z_t \mid x_t) \overline{bel}(x_t)$
5. endfor
6. Return bel(x_t)

Bayes Filter Reminder

- Prediction

$$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\, bel(x_{t-1})\, dx_{t-1}$$

- Correction

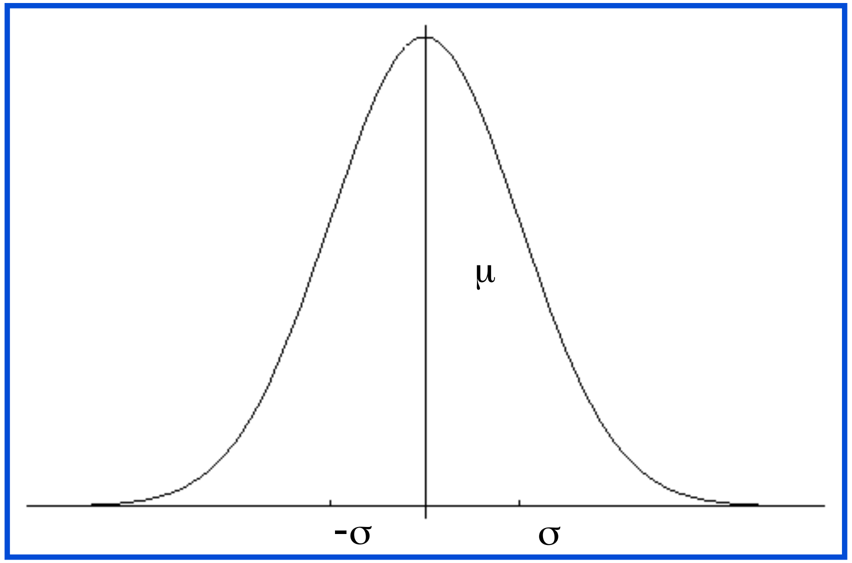
$$bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)$$

Gaussians

$$p(x) \sim N(\mu, \sigma^2):$$

$$p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

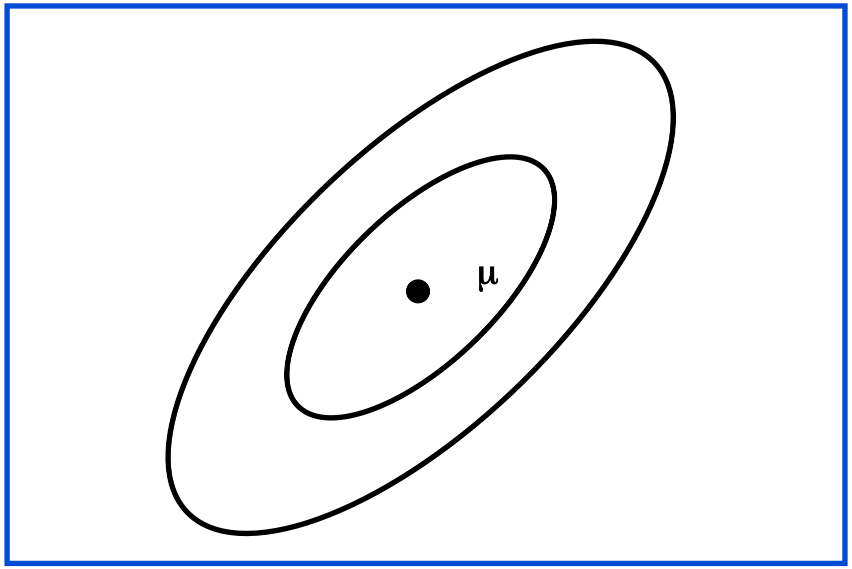
Univariate



$$p(\mathbf{x}) \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}):$$

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}|\boldsymbol{\Sigma}|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^t \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})}$$

Multivariate



Properties of Gaussians

$$\left.\begin{array}{l} X \sim N(\mu, \sigma^2) \\ Y = aX + b \end{array}\right\} \quad \Rightarrow \quad Y \sim N(a\mu + b, a^2\sigma^2)$$

$$\left.\begin{array}{l} X_1 \sim N(\mu_1, \sigma_1^{\ 2}) \\ X_2 \sim N(\mu_2, \sigma_2^{\ 2}) \end{array}\right\} \Rightarrow p(X_1) \cdot p(X_2) \sim N\left(\frac{\sigma_2^{\ 2}}{\sigma_1^{\ 2} + \sigma_2^{\ 2}} \mu_1 + \frac{\sigma_1^{\ 2}}{\sigma_1^{\ 2} + \sigma_2^{\ 2}} \mu_2, \quad \frac{1}{\sigma_1^{\ -2} + \sigma_2^{\ -2}} \right)$$

Multivariate Gaussians

$$\left.\begin{array}{l} X \sim N(\mu, \Sigma) \\ Y = AX + B \end{array}\right\} \Rightarrow \quad Y \sim N(A\mu + B, A\Sigma A^T)$$

$$\left.\begin{array}{l} X_1 \sim N(\mu_1, \Sigma_1) \\ X_2 \sim N(\mu_2, \Sigma_2) \end{array}\right\} \Rightarrow p(X_1) \cdot p(X_2) \sim N\left(\frac{\Sigma_2}{\Sigma_1 + \Sigma_2} \mu_1 + \frac{\Sigma_1}{\Sigma_1 + \Sigma_2} \mu_2, \quad \frac{1}{\Sigma_1^{-1} + \Sigma_2^{-1}} \right)$$

- We stay in the "Gaussian world" as long as we start with Gaussians and perform only linear transformations.

Kalman Filter

Estimates the state x of a discrete-time controlled process that is governed by the linear stochastic difference equation

$$\boxed{x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t}$$

with a measurement

$$\boxed{z_t = C_t x_t + \delta_t}$$

Components of a Kalman Filter

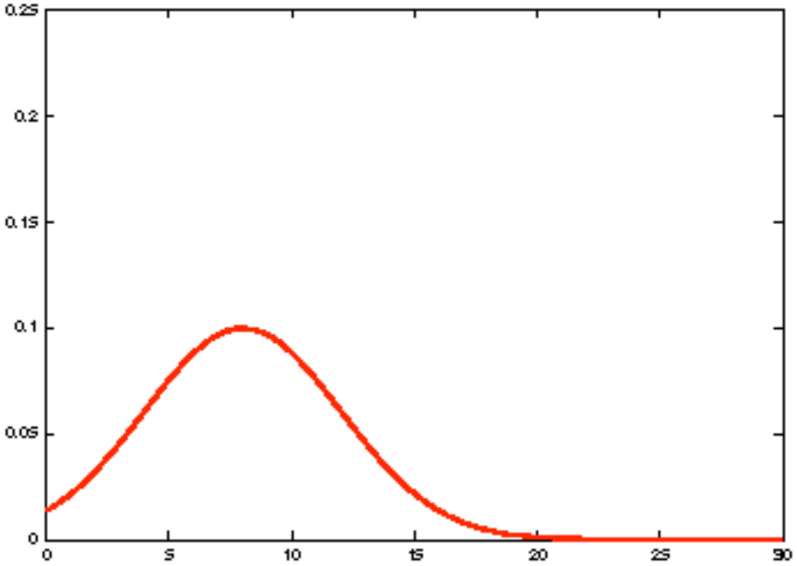
A_t Matrix (nxn) that describes how the state evolves from $t\text{-}1$ to t without controls or noise.

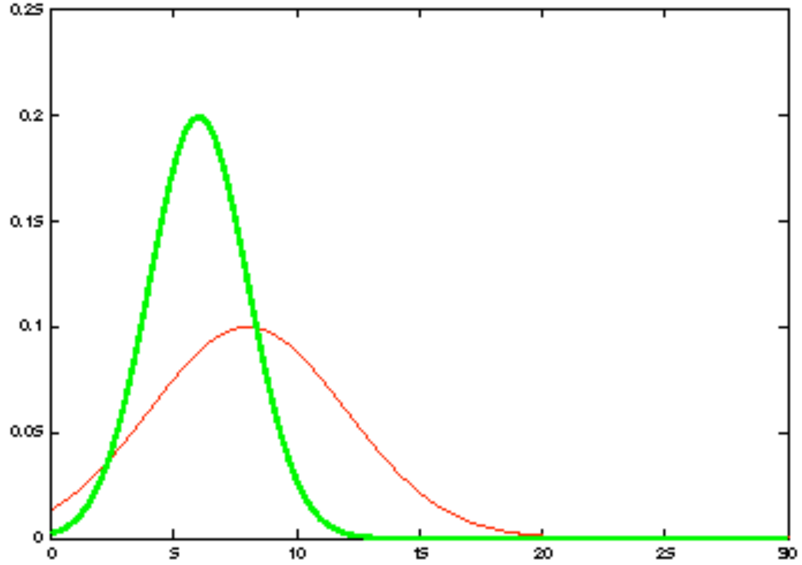
B_t Matrix (nxl) that describes how the control u_t changes the state from $t\text{-}1$ to t .

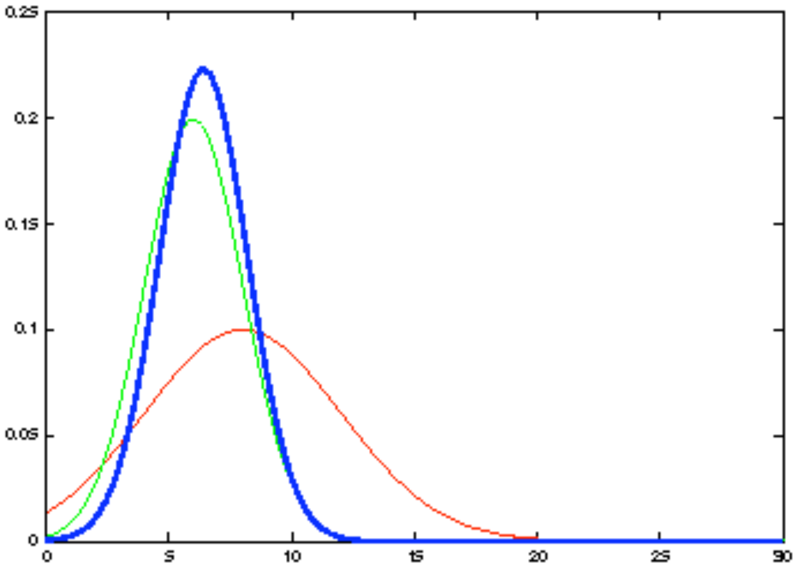
C_t Matrix (kxn) that describes how to map the state x_t to an observation z_t.

ε_t δ_t Random variables representing the process and measurement noise that are assumed to be independent and normally distributed with covariance R_t and Q_t respectively.

Kalman Filter Updates in 1D



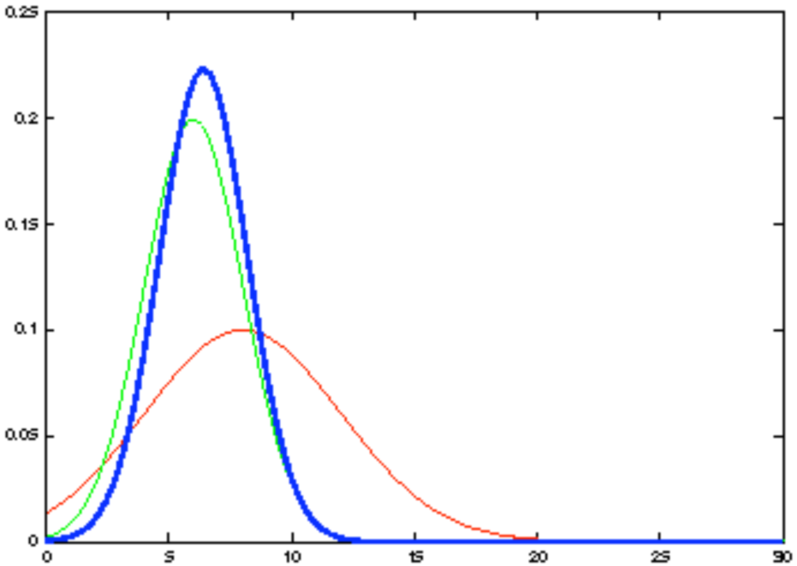




Kalman Filter Updates in 1D

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - \bar{\mu}_t) \\ \sigma_t^2 = (1 - K_t)\bar{\sigma}_t^2 \end{cases} \quad \text{with} \quad K_t = \frac{\bar{\sigma}_t^2}{\bar{\sigma}_t^2 + \bar{\sigma}_{obs,t}^2}$$

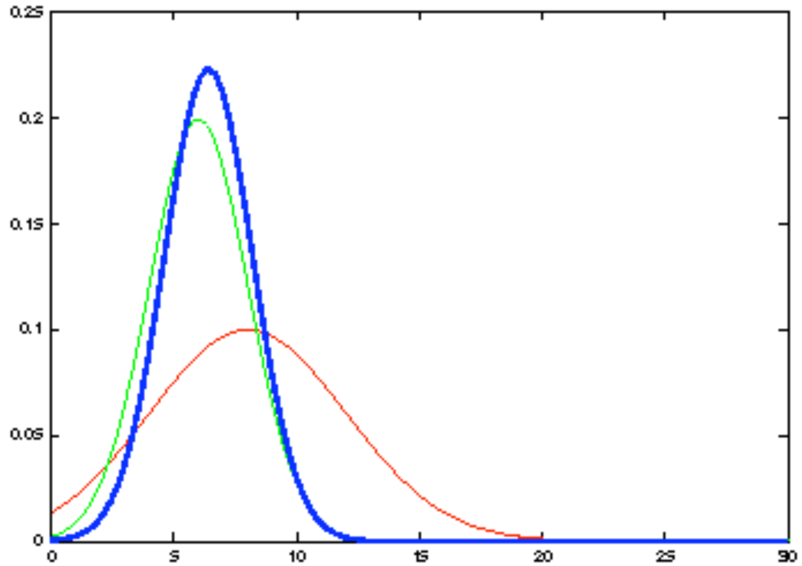
$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t\bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t)\bar{\Sigma}_t \end{cases} \quad \text{with} \quad K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$

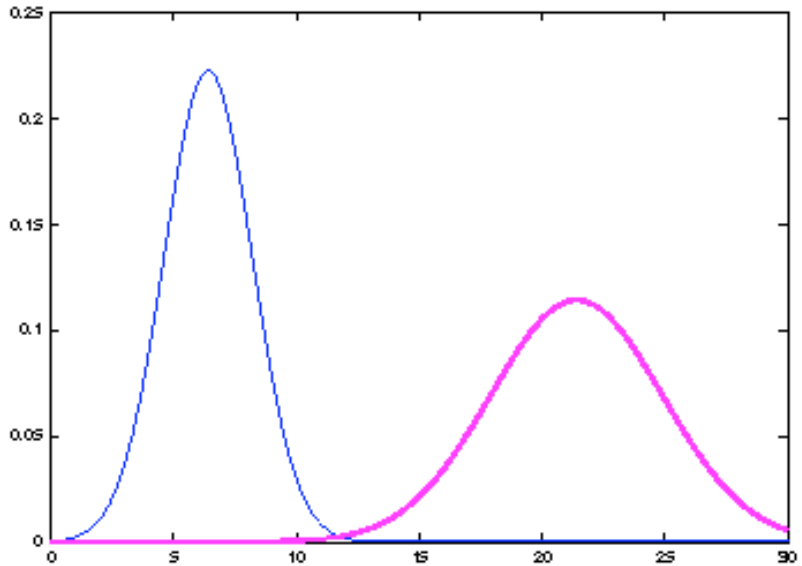


Kalman Filter Updates in 1D

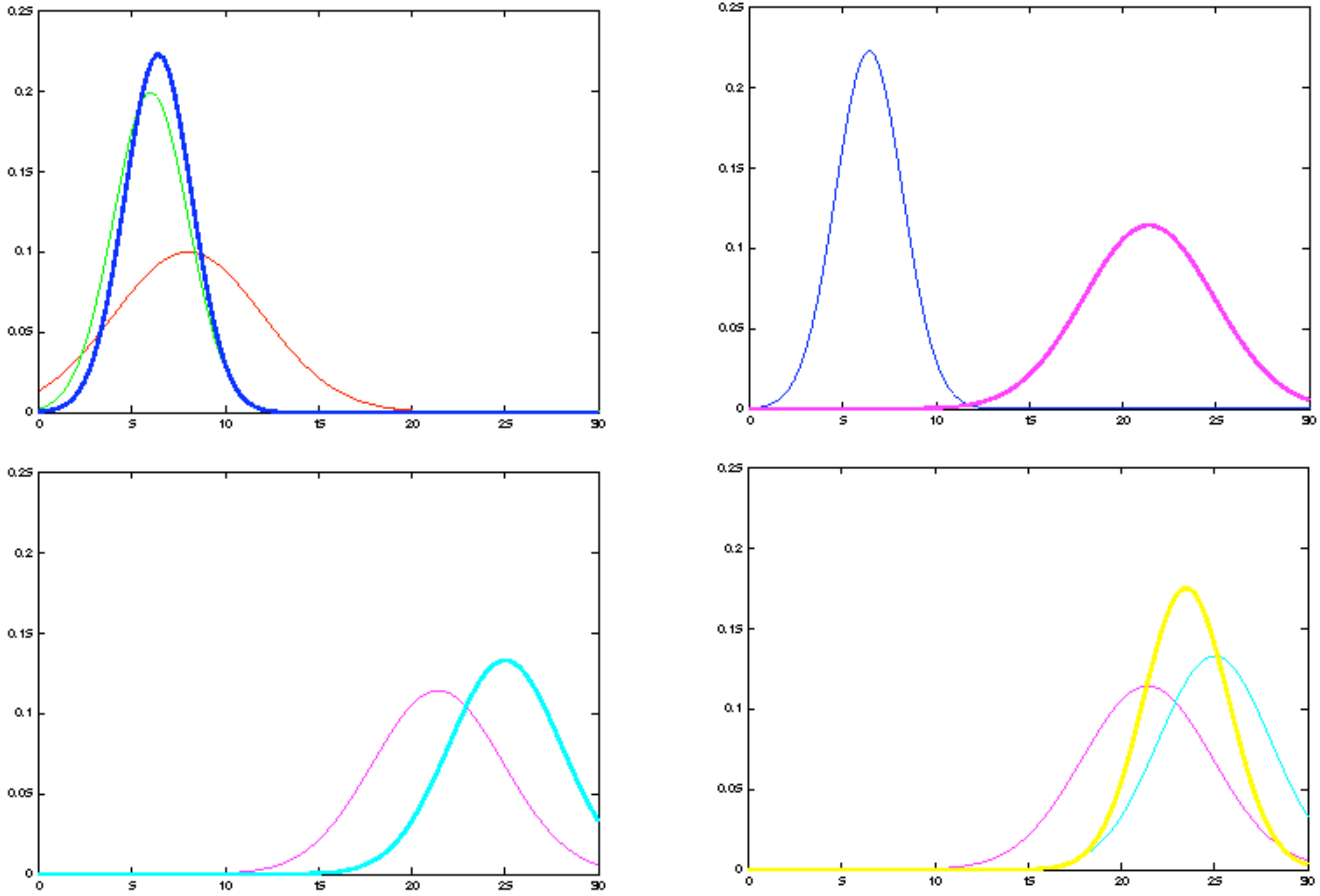
$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_t^2 + \sigma_{act,t}^2 \end{cases}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases}$$





Kalman Filter Updates



Linear Gaussian Systems: Initialization

- Initial belief is normally distributed:

$$\boxed{bel(x_0) = N\left(x_0; \mu_0, \Sigma_0\right)}$$

Linear Gaussian Systems: Dynamics

- Dynamics are linear function of state and control plus additive noise:

$$\boxed{x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t}$$

$$\boxed{p(x_t \mid u_t, x_{t-1}) = N\left(x_t; A_t x_{t-1} + B_t u_t, R_t\right)}$$

$$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1}) \qquad bel(x_{t-1})\, dx_{t-1}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sim N\left(x_t; A_t x_{t-1} + B_t u_t, R_t\right) \quad \sim N\left(x_{t-1}; \mu_{t-1}, \Sigma_{t-1}\right)$$

Linear Gaussian Systems: Dynamics

$$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1}) \qquad bel(x_{t-1})\, dx_{t-1}$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$\sim N\left(x_t; A_t x_{t-1} + B_t u_t, R_t\right) \quad \sim N\left(x_{t-1}; \mu_{t-1}, \Sigma_{t-1}\right)$$

$$\Downarrow$$

$$\overline{bel}(x_t) = \eta \int \exp\left\{-\frac{1}{2}(x_t - A_t x_{t-1} - B_t u_t)^T R_t^{-1}(x_t - A_t x_{t-1} - B_t u_t)\right\}$$

$$\exp\left\{-\frac{1}{2}(x_{t-1} - \mu_{t-1})^T \Sigma_{t-1}^{-1}(x_{t-1} - \mu_{t-1})\right\} dx_{t-1}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases}$$

Linear Gaussian Systems: Observations

- Observations are linear function of state plus additive noise:

$$\boxed{z_t = C_t x_t + \delta_t}$$

$$\boxed{p(z_t \mid x_t) = N\left(z_t; C_t x_t, Q_t\right)}$$

$$\boxed{\begin{array}{cccc} bel(x_t) = & \eta & p(z_t \mid x_t) & \overline{bel}(x_t) \\ & & \Downarrow & \Downarrow \\ & & \sim N\left(z_t; C_t x_t, Q_t\right) & \sim N\left(x_t; \bar{\mu}_t, \bar{\Sigma}_t\right) \end{array}}$$

Linear Gaussian Systems: Observations

$$bel(x_t) = \eta \quad p(z_t \mid x_t) \qquad \overline{bel}(x_t)$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sim N\left(z_t; C_t x_t, Q_t\right) \qquad \sim N\left(x_t; \bar{\mu}_t, \bar{\Sigma}_t\right)$$

$$\Downarrow$$

$$bel(x_t) = \eta \exp\left\{-\frac{1}{2}(z_t - C_t x_t)^T Q_t^{-1}(z_t - C_t x_t)\right\} \exp\left\{-\frac{1}{2}(x_t - \bar{\mu}_t)^T \bar{\Sigma}_t^{-1}(x_t - \bar{\mu}_t)\right\}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t \bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t)\bar{\Sigma}_t \end{cases} \qquad \text{with} \quad K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$

Kalman Filter Algorithm

1. Algorithm **Kalman_filter**(μ_{t-1}, Σ_{t-1}, u_t, z_t):

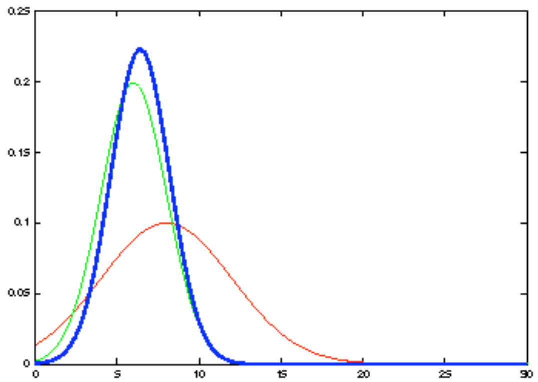
2. Prediction:
3. $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$
4. $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$

5. Correction:
6. $K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$
7. $\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$
8. $\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$

9. Return μ_t, Σ_t

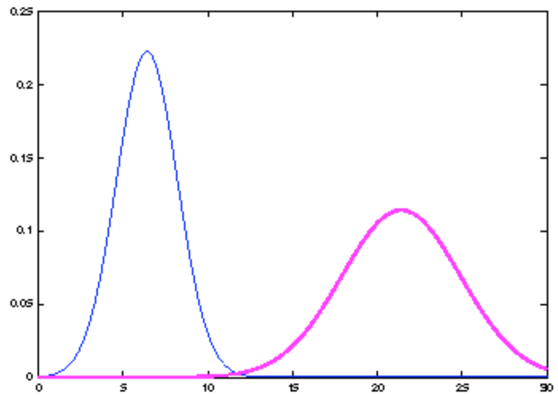
The Prediction-Correction-Cycle

Prediction

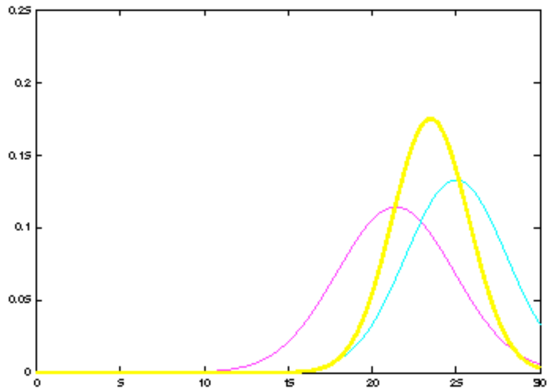


$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_t^2 + \sigma_{act,t}^2 \end{cases}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases}$$

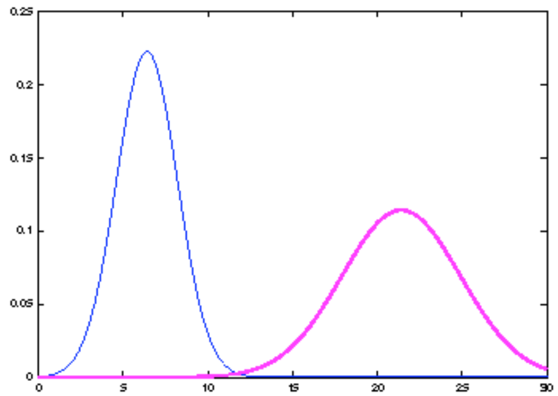


The Prediction-Correction-Cycle



$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - \bar{\mu}_t) \\ \sigma_t^2 = (1 - K_t)\bar{\sigma}_t^2 \end{cases}, K_t = \frac{\bar{\sigma}_t^2}{\bar{\sigma}_t^2 + \bar{\sigma}_{obs,t}^2}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t\bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t)\bar{\Sigma}_t \end{cases}, K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$



Correction

The Prediction-Correction-Cycle

Prediction

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_t^2 + \sigma_{act,t}^2 \end{cases}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - \bar{\mu}_t) \\ \sigma_t^2 = (1 - K_t)\bar{\sigma}_t^2 \end{cases}, K_t = \frac{\bar{\sigma}_t^2}{\bar{\sigma}_t^2 + \bar{\sigma}_{obs,t}^2}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t \bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t)\bar{\Sigma}_t \end{cases}, K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$

Correction

Kalman Filter Summary

- Highly efficient: Polynomial in measurement dimensionality *k* and state dimensionality *n*:

$$O(k^{2.376} + n^2)$$

- Optimal for linear Gaussian systems!

- Most robotics systems are nonlinear!

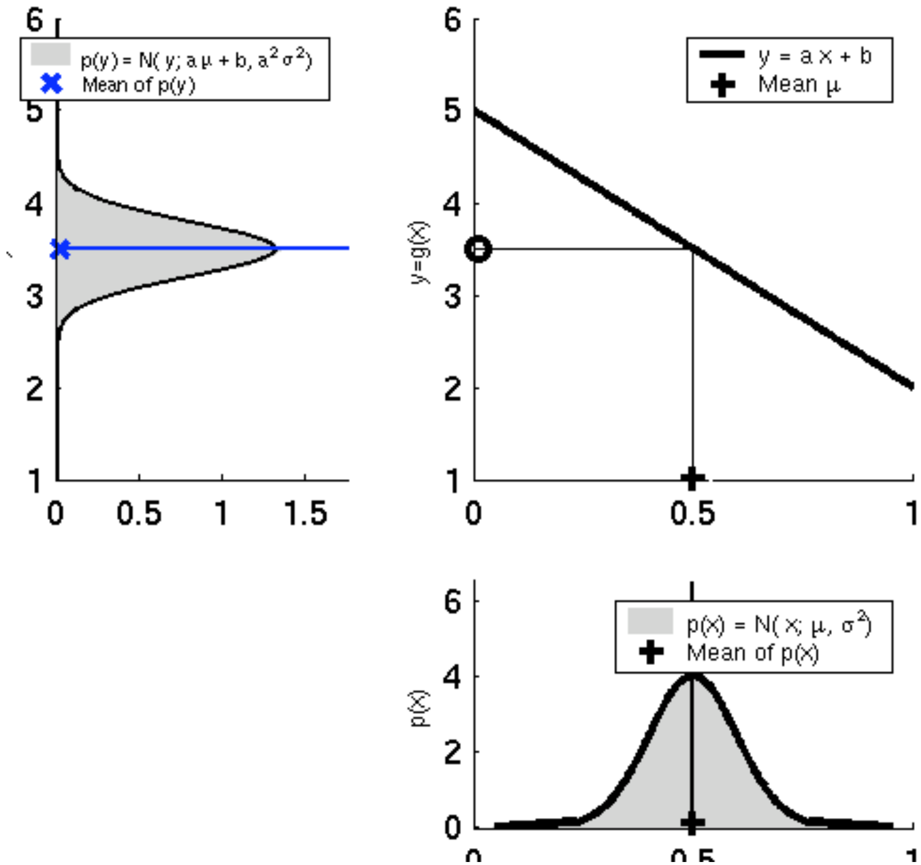
Nonlinear Dynamic Systems

- Most realistic robotic problems involve nonlinear functions

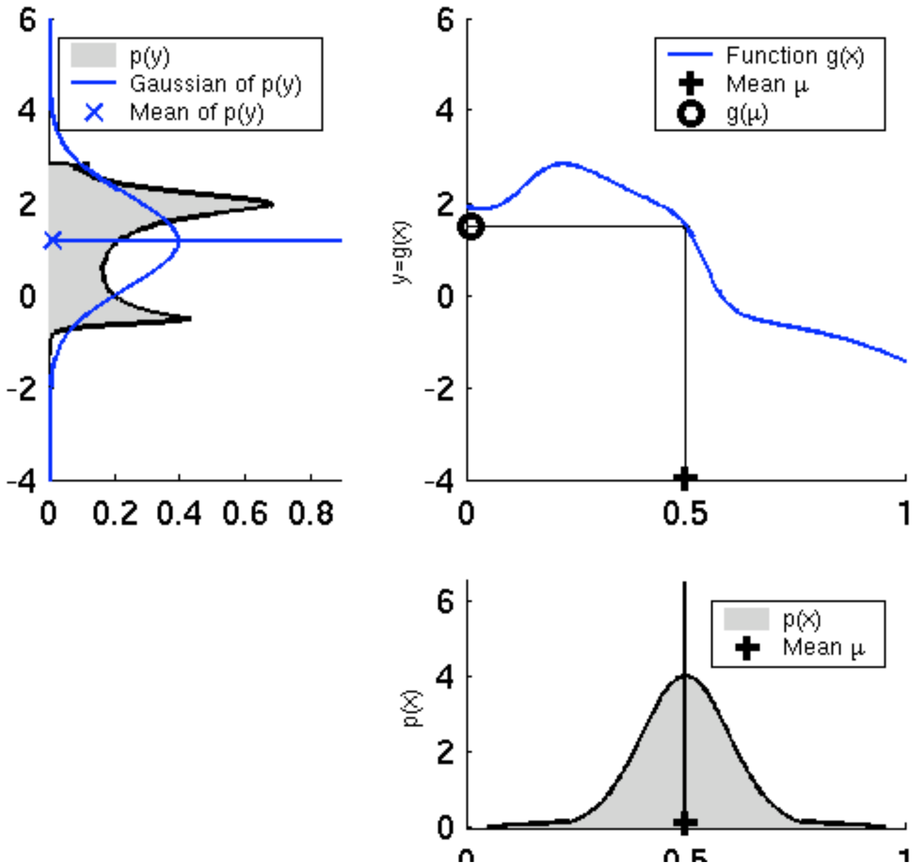
$$\boxed{x_t = g(u_t, x_{t-1})}$$

$$\boxed{z_t = h(x_t)}$$

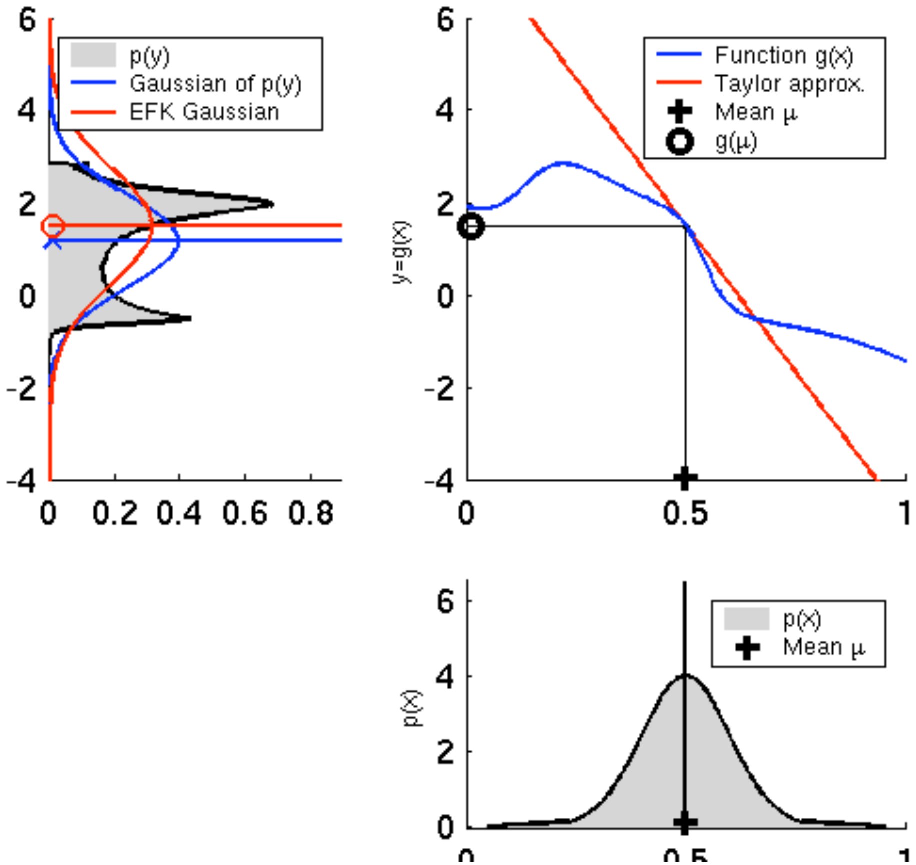
Linearity Assumption Revisited



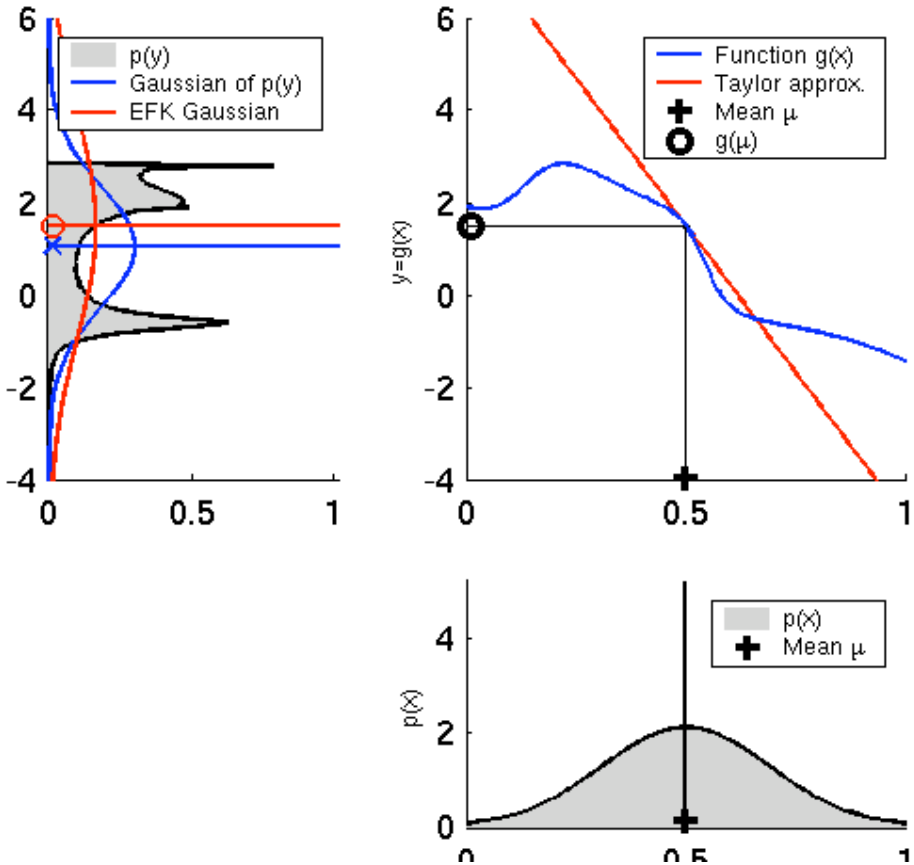
Non-linear Function



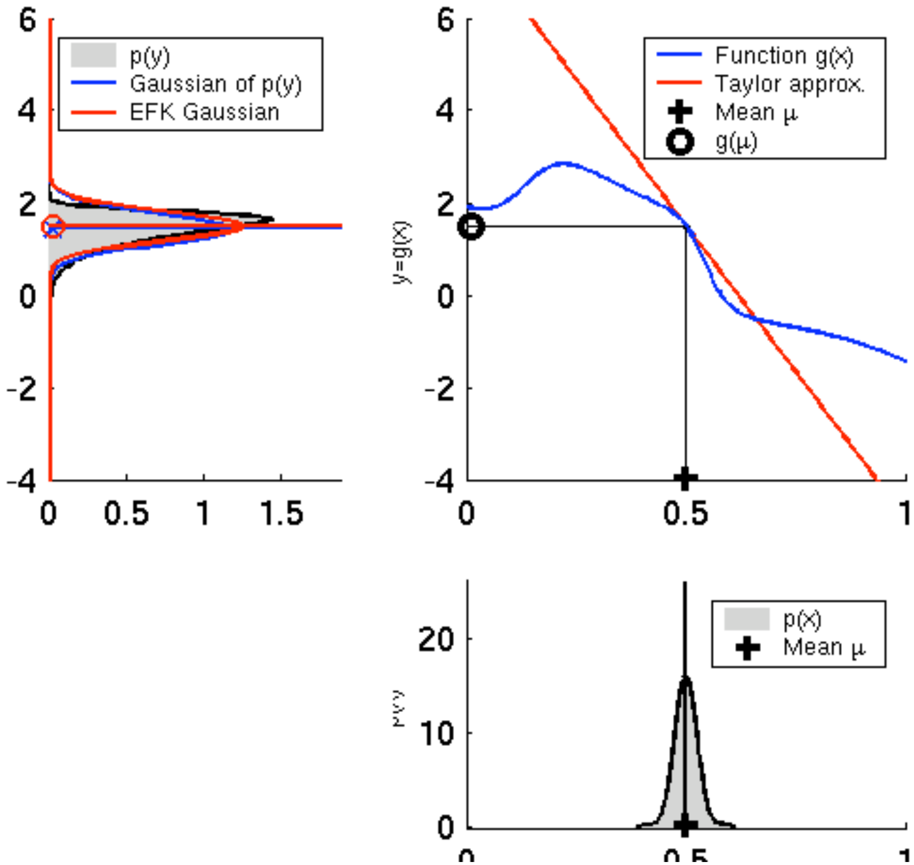
EKF Linearization (1)



EKF Linearization (2)



EKF Linearization (3)



EKF Linearization: First Order Taylor Series Expansion

- Prediction:

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}} (x_{t-1} - \mu_{t-1})$$

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + G_t \, (x_{t-1} - \mu_{t-1})$$

- Correction:

$$h(x_t) \approx h(\overline{\mu}_t) + \frac{\partial h(\overline{\mu}_t)}{\partial x_t} (x_t - \overline{\mu}_t)$$

$$h(x_t) \approx h(\overline{\mu}_t) + H_t \, (x_t - \overline{\mu}_t)$$

EKF Algorithm

1. **Extended_Kalman_filter**(μ_{t-1}, Σ_{t-1}, u_t, z_t):

2. Prediction:

3. $\bar{\mu}_t = g(u_t, \mu_{t-1})$ ⟵ $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$

4. $\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + R_t$ ⟵ $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$

5. Correction:

6. $K_t = \bar{\Sigma}_t H_t^T (H_t \bar{\Sigma}_t H_t^T + Q_t)^{-1}$ ⟵ $K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$

7. $\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$ ⟵ $\mu_t = \bar{\mu}_t + K_t(z_t - C_t \bar{\mu}_t)$

8. $\Sigma_t = (I - K_t H_t)\bar{\Sigma}_t$ ⟵ $\Sigma_t = (I - K_t C_t)\bar{\Sigma}_t$

9. Return μ_t, Σ_t

$$H_t = \frac{\partial h(\bar{\mu}_t)}{\partial x_t} \qquad G_t = \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}}$$