BEST-REPLY SEARCH IN MULTI-PLAYER CHESS

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Master Thesis 12-13

Thesis submitted in partial fulfilment
of the requirements for the degree of
Master of Science of Artificial Intelligence
at the Faculty of Humanities and Sciences
of Maastricht University

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July 2012

Preface

This report is my master thesis. It describes the performed research at the Department of Knowledge Engineering at Maastricht University. In this thesis, four variants of the multi-player search algorithm *best-reply search* are proposed. These proposed algorithms are tested in the domain of multi-player Chess.
I would like to thank both my supervisors dr. Maarten Schadd and dr. Mark Winands. Dr. Maarten Schadd was my supervisor during the first part of the thesis and gave me practical hints for the implementation of the existing algorithms and search enhancements. Especially, he indicated common sources of errors which probably reduced the time for debugging the algorithms. During the last part of the thesis, dr. Mark Winands was my daily supervisor. His knowledge about programming Artificial Intelligence in Chess was helpful to increase the performance of my program for playing multi-player Chess. Additionally, his input and corrections were useful to improve this report. I also would like to thank my fellow student Michael Gras for a lot of interesting discussions about the proposed *best-reply* variants and how they perform in different domains. Finally, I would like to thank my parents who supported me during this thesis and the whole studies.
I got the idea to propose variants of the *best-reply search* during the course "Intelligent Search Techniques" given as part of the Master of Artificial Intelligence at the Department of Knowledge Engineering. The *best-reply search*, recently invented at this Department, was mentioned in this course as a promising search algorithm for multi-player games.

Markus Esser
Maastricht, July 2012

Abstract

Computers have been used to play board games, especially Chess, since their earliest days. Researchers mainly concentrated on building strong programs for playing two-player games. Therefore much research can still be done in the area of multi-player games. This thesis is about search algorithms in multi-player games. A program for playing multi-player Chess is build as a test domain to perform research on it. Multi-player Chess is a Chess variant for four players. As regular Chess, multi-player Chess is a capture game with perfect information. Although Monte-Carlo Tree Search nowadays has become popular in the area of games, this thesis focusses on evaluation function based search. Recently, a new promising search algorithm for multi-player games, called *best-reply search*, was proposed by Schadd and Winands (2011). In this algorithm, only the opponent with the strongest counter-move is allowed to play a move. The other opponents pass their turn. Although *best-reply* performs well in the games tested so far, it has conceptual drawbacks. The main drawback of *best-reply search* is that the search can lead to officially unreachable and sometimes illegal board positions. The concern of this thesis is to overcome these drawbacks and thus, to improve the performance of *best-reply search*. Therefore the problem statement is:
How can the best-reply search be modified to outperform the paranoid algorithm in complex (capturing) multi-player games?
To introduce the test domain in which the research is performed, this thesis starts with an explanation of the rules of multi-player Chess. Afterwards, the existing search algorithms and their enhancements are described. Next, new search algorithms based on the *best-reply search* are proposed. These algorithms are explained and analyzed. The main idea to overcome the drawbacks of *best-reply* is to let the opponents play the best move regarding the static move ordering instead of passing as done in *best-reply search*. After a description of the evaluation function, which is required to gain reasonable play in the area of multi-player Chess, the proposed algorithms are tested against the existing ones. The experiments show that one of the proposed algorithms, namely $BRS_{1,C-1}$, is able to outperform the existing algorithms in the domain of multi-player Chess. In $BRS_{1,C-1}$, the opponent with the strongest counter-move is allowed to perform a search while the other opponents play the best move regarding the static move ordering. Because of the strong performance in multi-player Chess, $BRS_{1,C-1}$ is a promising search algorithm. Further tests have to be performed to detect whether it also works well in other domains than multi-player Chess.

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Chapter 1

Introduction

This chapter first gives a brief overview about games and Artificial Intelligence (AI) in Section 1.1. An introduction to AI in multi-player games is given in Section 1.2. The problem statement and research questions of this thesis are introduced in Section 1.3. Finally, an outline of the thesis is given in Section 1.4.

1.1 Games and Artificial Intelligence (AI)

Costikyan (1994) defined a game as "a form of art in which participants, termed players, make decisions in order to manage resources through game tokens in the pursuit of a goal". The players in a game can have perfect information like in Chess or imperfect information like in card games (e.g., Poker), where the players cannot see the cards of their opponents. Another important feature to categorize games is whether there is chance involved, e.g., rolling a dice, or not. In all games, the players have to make optimal decisions to win the game. This thesis is about search algorithms for making optimal decisions in multi-player Chess. Multi-player Chess is a multi-player game with perfect information without chance. Since the earliest days of computers, they have been used to play board games, especially Chess (Shannon, 1950; Turing, 1953). Board games have a special interest for researchers because the rules are well defined and therefore can easily be modelled in a computer program. In quite some two-player games, e.g., Chess (Hsu, 2002) and Checkers (Schaeffer and Lake, 1996), programs are nowadays able to play successfully against the best human players. Research has been mainly concentrated on two-player games and thus much progress can be made in the area of multi-player games.

1.2 AI in Multi-Player Games

Multi-player games are designed to be played by more than two players. Nevertheless some multi-player games can also be played by two players, like Poker or Chinese Checkers. Multi-player games can be cooperative. This means that players form groups, called coalitions. The game is a competition between these coalitions instead of a competition between individual players. Multi-player Chess is one of the games which can be played cooperative or non-cooperative. In this thesis, the non-cooperative version is used, which means that all players play against each other.
There exist several approaches to create a strong AI for multi-player games. Schadd (2011) gives a good summarization of these approaches. Tree-search methods for coalition forming (Sturtevant and Korf, 2000) and Monte-Carlo methods (Sturtevant, 2008) have been applied to Chinese Checkers. For the well-known game Monopoly[1], Markov chains (Ash and Bishop, 1972) can be applied and for learning strategies in this game, evolutionary algorithms can be used (Frayn, 2005). For other games, Bayesian networks can be useful. In games like Poker, opponent modelling is quite important to build a strong program (Billings *et al.*, 1998).
The program in this thesis makes decisions by searching. It builds a tree structure where the edges represent the moves and the nodes represent the board positions. In general, the search can be based

[1] Monopoly® is a registered trademark of Hasbro, Inc. All rights reserved.

on an evaluation function, as explained in Subsection 1.2.1, or on randomized explorations of the search space, explained in Subsection 1.2.2. Only the first approach is used in this thesis.

1.2.1 Evaluation Function Based Search

In a search based on an evaluation function, the AI requires a decision rule (Chapter 3) and an evaluation function (Chapter 5) to find a good move in the game tree. In the following, three decision rules for multi-player games are briefly introduced. These decision rules are used in this thesis and are explained in more detail in Subsection 3.3.1 (max^n), 3.3.2 (*paranoid*) and 3.3.3 (*best-reply*).

- **Max^n**
  In max^n (Luckhart and Irani, 1986), the root player assumes that each player tries to maximize his own score.

- **Paranoid**
  In *paranoid* (Sturtevant and Korf, 2000), the root player assumes that all his opponents build a coalition against him. Therefore this player assumes that each opponent in the search tries to minimize the value of the root player. For the most multi-player games, *paranoid* outperforms max^n, because the pruning techniques available for *paranoid* are much more effective. Nevertheless, the *paranoid* decision rule has two disadvantages.

  1. It is not possible to explore many moves of the root player in a complex multi-player game and thus, there is no long-term planning (Schadd and Winands, 2011). This problem is even larger for max^n where pruning is not as powerful as the $\alpha\beta$-pruning for *paranoid*.
  2. Because of the unrealistic assumption that all opponent players form a coalition against the root player, suboptimal play can occur (Sturtevant and Korf, 2000).

- **Best-Reply**
  Best-reply search (Schadd and Winands, 2011) only allows the opponent with the strongest counter move to perform a move. The other opponents pass their move.

1.2.2 Monte-Carlo Tree Search (MCTS)

Monte-Carlo Tree Search (MCTS) (Coulom, 2007; Kocsis and Szepesvári, 2006) does not build a game tree of a fixed depth and does not use an evaluation function to assign values to the leaf nodes as the previous search algorithms. Instead, it is a best-first algorithm based on randomized explorations of the search space (Chaslot *et al.*, 2008a). "Using the results of previous explorations, the algorithm gradually grows a game tree in memory, and successively becomes better at accurately estimating the values of the most promising moves" (Chaslot, Winands, and Van den Herik, 2008b). Figure 1.1 illustrates the four phases of MCTS.

1. **Selection**
   In this phase, the tree is traversed from the root node until it selects a node not added to the tree so far.

2. **Expansion**
   In this phase, the node selected in the previous phase is added to the tree.

3. **Simulation**
   In this phase, moves are played until the end of the game is reached. The result of this search can be a win, a draw or a loss for the root player.

4. **Backpropagation**
   In this phase, the result of the simulation is propagated through the tree.

MCTS has become popular in the area of games (Browne *et al.*, 2012). Examples for two-player games where it has successfully been applied are Go (Chaslot *et al.*, 2008a; Coulom, 2007), Amazons (Lorentz, 2008), Lines of Action (Winands and Björnsson, 2010) and Hex (Cazenave and Saffidine, 2009). In the multi-player games Chinese Checkers (Sturtevant, 2008), multi-player Go (Cazenave, 2008) and Focus

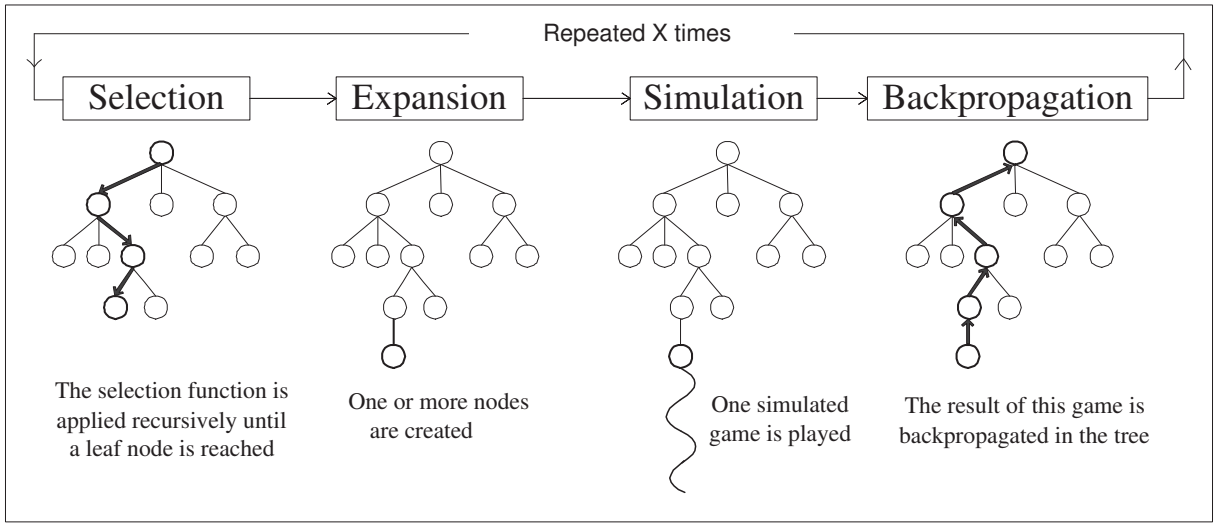


Figure 1.1: Four phases of MCTS; Figure taken from Chaslot *et al.* (2008a)

(Nijssen and Winands, 2011), MCTS was also applied successfully. In Chess and other games with short tactical goals, a traditional minimax based approach equipped with an evaluation function is still more successful than MCTS. Ramanujan, Sabharwal, and Selman (2010) pointed out that MCTS is not able to detect traps. The presence or absence of traps in a game is an indication whether MCTS can successfully be applied or not. It can happen that MCTS plays bad moves in such games because it often assigns a value to a trap move that is quite close to the value assigned to the best move. In general, traps often occur in games where wins and losses are defined through a sudden-death property (Ramanujan *et al.*, 2010). A sudden-death is an abrupt end of a game by the creation of one of a predefined set of patterns (Allis, 1994). In the domain of multi-player Chess, the capture of the King is such a sudden-death property and therefore MCTS is not expected to perform as good as the minimax-based search algorithms equipped with an evaluation function. Thus, MCTS is not considered in this thesis.

1.3 Problem Statement and Research Questions

There is a requirement for a new search algorithm because the well-known *paranoid* algorithm (Sturtevant and Korf, 2000) for multi-player games has two known disadvantages, the absence of long-term planning and the unrealistic assumption that everybody is against you. Thus, the goal of this thesis is to improve the idea of the *best-reply search* (Schadd and Winands, 2011). This is tested in the domain of multi-player Chess, a perfect information multi-player game.
The problem statement is:

How can the best-reply search be modified to outperform the paranoid algorithm in complex (capturing) multi-player games?

The following two research questions to answer this problem are:

1. Is the best-reply search better than the well-known paranoid and max^n algorithms in the domain of non-cooperative multi-player Chess?

The first step to answer this question is to implement an efficient framework for playing multi-player Chess. Three search algorithms for multi-player games, namely max^n, *paranoid* and *best-reply*, have to be implemented with all the known search enhancements like pruning, move ordering etc. (see Chapter 3). To let the players play on a high level, an evaluation function has to be implemented (Chapter 5).

Finally, the developed framework can be used to check the performance of the *best-reply search* against the two other algorithms.

2. Can we find variants of the best-reply search that are even better?

To answer this question first the weaknesses of the *best-reply search* has to be discussed. Ideas to overcome these weaknesses are proposed (see Chapter 4) and implemented. The main idea proposed in this thesis is to let the passing opponents play the best move from the static move ordering. Further, the number of searching players can be varied. Finally, experiments have to be executed to find out which variant of the *best-reply search* is the strongest one and how does it perform against the *paranoid* and max^n algorithms in the domain of multi-player Chess.

1.4 Outline of the Thesis

Here a brief outline of the thesis is given.

- Chapter 1 gives an introduction to artificial intelligence in multi-player games. Further, the problem statement and research questions are stated.
- Chapter 2 describes the game of multi-player Chess.
- Chapter 3 explains the search techniques used in the program.
- Chapter 4 describes the proposed *best-reply search* variants.
- Chapter 5 discusses the evaluation function required to assign values to the leaf nodes of the game tree.
- Chapter 6 explains the performed experiments and presents the corresponding results.
- Chapter 7 presents the conclusion of this thesis and the future research.

Chapter 2

Multi-Player Chess

This chapter describes the game of multi-player Chess which is used as test domain in this thesis. Section 2.1 gives an introduction to regular Chess. Section 2.2 explains multi-player Chess itself. In Section 2.3, the complexity of multi-player Chess is computed.

2.1 Two-Player Chess

An insight into the history of Chess is given in Subsection 2.1.1. The rules of regular Chess are explained from Subsection 2.1.2 through to 2.1.5

2.1.1 History

The earliest predecessors of Chess originated before the 6^{th} century in India. From India, the game spread to Persia where they started calling "Shāh!" (Persian for "King!") when threaten the opponent's King, and "Shāh Māt" (Persian for "the King is helpless") when the King can not escape from attack. These exclamations persisted in Chess, e.g., nowadays in Germany the exclamations are still "Schach!" and "Schach Matt!". When the Arabs conquered Persia, Chess was taken up by the Muslim world and subsequently spread to Europe and Russia at the end of the first millennium. In the 15^{th} century, the Bishop's and Queen's moves were changed and Chess became close to its current form. Also the first writings about the theory of playing Chess appeared in that century. In the 19^{th} century many Chess clubs, Chess books and Chess journals appeared and the first modern Chess tournament play began. The first World Chess Championship was held in 1886. Position analysis became also popular during this time.

2.1.2 Board and Initial Position

Chess is played on an 8×8 board. The players are called White and Black. Both players start with 16 pieces as shown in Figure 2.1. Each player has eight Pawns ♙, two Rooks ♖, two Knights ♘, two Bishops ♗, one Queen ♕ and one King ♔.

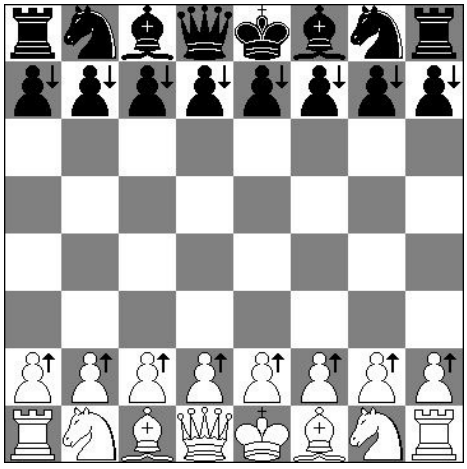
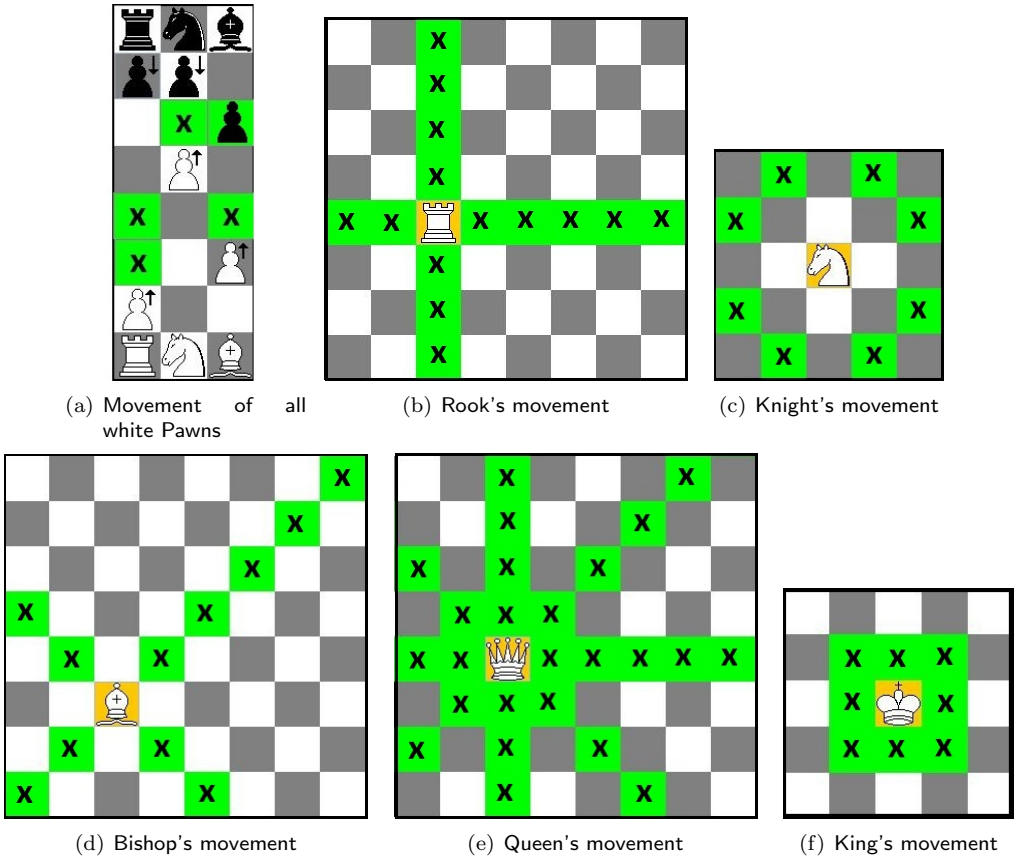


Figure 2.1: Initial Position in Chess

2.1.3 Movement

The players move alternately, starting with White. Moving means that the player moves one of his pieces to an empty square or a square with an opponent piece on it. If the square has an opponent's piece on it, this piece gets removed from the board. In general it is not allowed to jump over other pieces. Only the Knight is allowed to do this. Each kind of piece has its own rules how it is allowed to move. These individual moving rules are as follows.



(a) Movement of all white Pawns (b) Rook's movement (c) Knight's movement

(d) Bishop's movement (e) Queen's movement (f) King's movement

Figure 2.2: Movement of all different pieces in Chess

Pawn ♙ ♟

A Pawn has two options to make a move. The first option is to move straight forward to the next square if this square is empty. The second option is to capture an opponent's piece with a diagonally forward move. When a Pawn has not been moved yet in the game, it is allowed to move two squares forward if not obstructed. All these movements are illustrated in Figure 2.2(a).

Rook ♖ ♜

The Rook can move orthogonally as far as possible (see Figure 2.2(b)).

Knight ♘ ♞

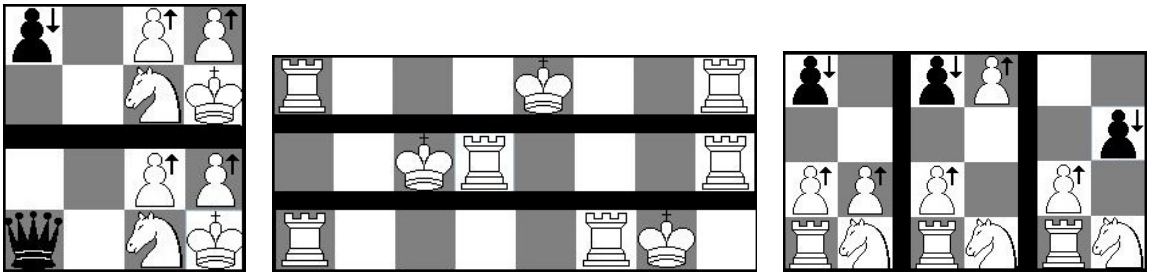
The Knight is allowed to jump over other pieces regardless whether it is an own or an opponent's piece. The moves shown in 2.2(c) are jumps two squares in one direction and one square in one of the perpendicular directions.

Bishop ♗ ♝

The Bishop is allowed to move in each diagonal direction as far as possible. This can be seen in Figure 2.2(d).

Queen ♕ ♛

The Queen is allowed to move in any direction as far as possible as demonstrated in Figure 2.2(e).



(a) A black Pawn (upper part) reaches the opposite of the board and promotes to a Queen (lower part)

(b) Castling, upper row: position before castling, middle row: position after long castling, lower row: position after short castling

(c) En Passant Move; position before double step, position after double step,position after en-passant move

Figure 2.3: The three special moves in Chess

King ♔♚

The King is allowed to make a move to every adjacent square as Figure 2.2(f) shows.

2.1.4 Special Moves

Additionally to the regular movement rules there are three special moves, called *promotion*, *castling* and *en-passant*.

Promotion

A Pawn promotes if it reaches the opposite side of the board. Promotion means that the Pawn gets replaced by a Rook, a Bishop, a Knight or a Queen. The player can choose to which of these pieces the Pawn promotes. The choice is not limited to the previously captured pieces (e.g., it is allowed to have two Queens on the board). A promotion to a black Queen ♛ is depicted in Figure 2.3(a).

Castling

The castling move is the only move where two own pieces are moved simultaneously, the King and a Rook. The King is allowed to move two squares towards the Rook. At the same time the Rook moves towards the King and jumps over him so that his position is next to the King. Figure 2.3(b) demonstrates this. The upper row in the figure shows the situation before the castling move is performed. In the middle row the situation after a long castling is mapped. This move is called long castling because the Rook ♖ moves three squares while in a short castling illustrated in the lower row the Rook ♖ only moves two squares. A castling move can only be made when the following three conditions are met:

1. Neither the King nor the Rook has been moved in the whole game.
2. Every square between the King and the Rook is empty.
3. Neither the Kings square nor any other square the King passes or ends up in can be reached by an opponent's move.

En-Passant

En-passant is a special Pawn capture which can only occur immediately after a player moves a Pawn two squares forward from its starting position. If an opponent's Pawn is next to the moved Pawn, the opponent has the option of capturing the Pawn "en passant" as if the moved Pawn had only moved one square. This special move is illustrated in Figure 2.3(c).

2.1.5 End of the Game

The sole objective of the game is to capture the opponent's King. A move that threatens to capture the King is called a check move. The opponent has to react to this move. If the opponent cannot play a move to prevent his King from being captured, he is checkmate and loses the game. The game can also end in a draw when one of the following three conditions are met:

1. No progression, which means that no piece has been captured and no Pawn has been moved in the last 50 moves.
2. An identical position, including the player on turn, occurs for the third time.
3. The player on turn has no legal move and is not in check. This situation is called Stalemate.

2.2 Multi-Player Chess

In this section the history of multi-player Chess is outlined in Subsection 2.2.1. The supplemental rules are revealed from Subsection 2.2.2 through to 2.2.7. The multi-player Chess rules used in this thesis are taken over from Tscheuschner (2005).

2.2.1 History

Chess variants for more than two players have been invented for more than two hundred years. The board size was increased, depending on the variant by adding two to four rows to each side. The pieces used for each player are in nearly all variants the same as for regular Chess. But there are many different rule variations of multi-player Chess. The most common form of play is cooperative (two vs. two) in which allied pieces cannot eliminate each other. The game is over when both opposing Kings are checkmated. Playing non-cooperative is generally regarded more difficult than team play. Each player can attack any of the other three players and vice versa. Once a player is checkmated, depending on the variant either the pieces of the checkmated player gets removed from the board, or the player that checkmated can use the remaining pieces during his turn. For more information about multi-player Chess variants and Chess variants in general see Pritchard (2001). Non-cooperative play with removing the pieces of the defeated opponent is used in this thesis.

2.2.2 Board and Initial Position

The multi-player board (Figure 2.4) used in this thesis consists of the same board as the original Chess board with additionally three rows on each side of the board. The pieces are placed as in two-player Chess with one exception. The King is always placed right of the Queen. The order of moving is clockwise beginning with White.

2.2.3 Bending Off

In multi-player Chess, it is possible to change the direction of a Pawn. This happens on the main diagonals. There, the player has the choice to either keep the Pawns direction or to change the direction in a way that the number of moves to promote stays the same. A Pawn can promote at each of a opponent's side. Figure 2.5(a) demonstrates the two possibilities the white Pawn ♙ on K4 has. In this case, it is not allowed to change the direction to the East because the number of moves to promote would decrease.

Figure 2.5(b) depicts all the possible capturing moves a Pawn can have. As in Figure 2.5(a), the white Pawn ♙ on K4 can keep its direction by moving forward or change its direction by moving west. In this situation the white Pawn ♙ on K4 has three additional possibilities to move by capturing an opponent's piece. At the upper right board the white Pawn ♙ has captured the blue Pawn ♟ on L5 and has kept its direction. In the lower left board the white Pawn ♙ has captured the black Bishop ♝ on J3 and has changed its direction to West. When the white Pawn ♙ captures the red Queen ♛ on J5 as shown at the lower right board the player can still decide which direction the Pawn ♙ has. The white Pawn ♙ could capture the red Queen ♛ by keeping its direction. Therefore the decision whether the Pawn ♙ bends off is delayed to its next movement.

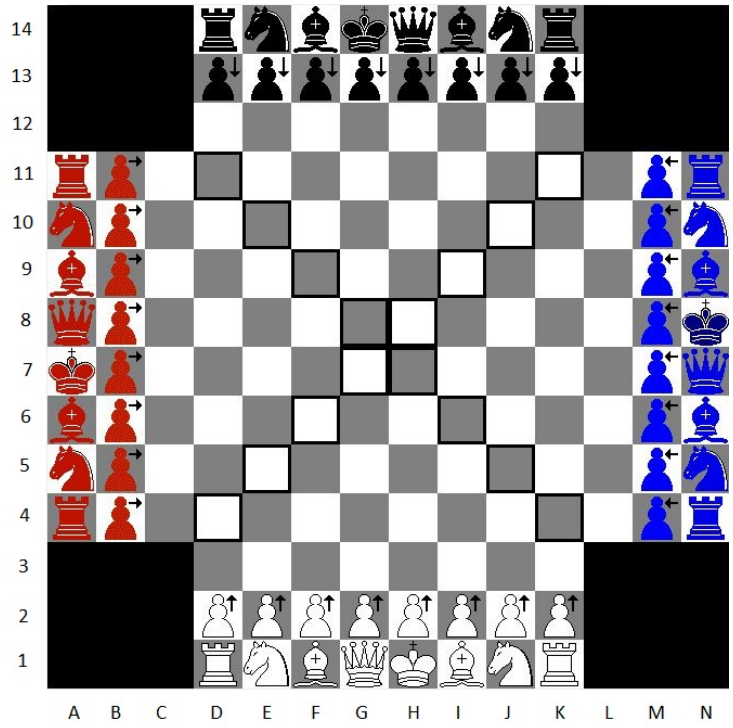
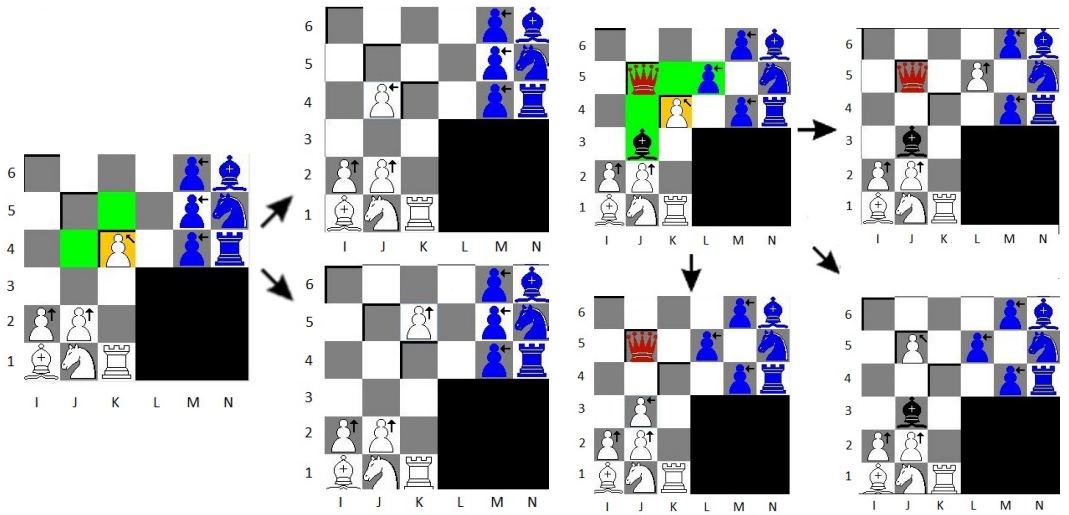


Figure 2.4: Initial Position in Four-Player Chess. The main diagonals, where the Pawns are allowed to bend off, run from D4 to K11 and from D11 to K4.



(a) The Pawn can keep its direction or change it.

(b) The Pawn has three possible capturing moves in this situation.

Figure 2.5: The Pawn's possibilities to bend off

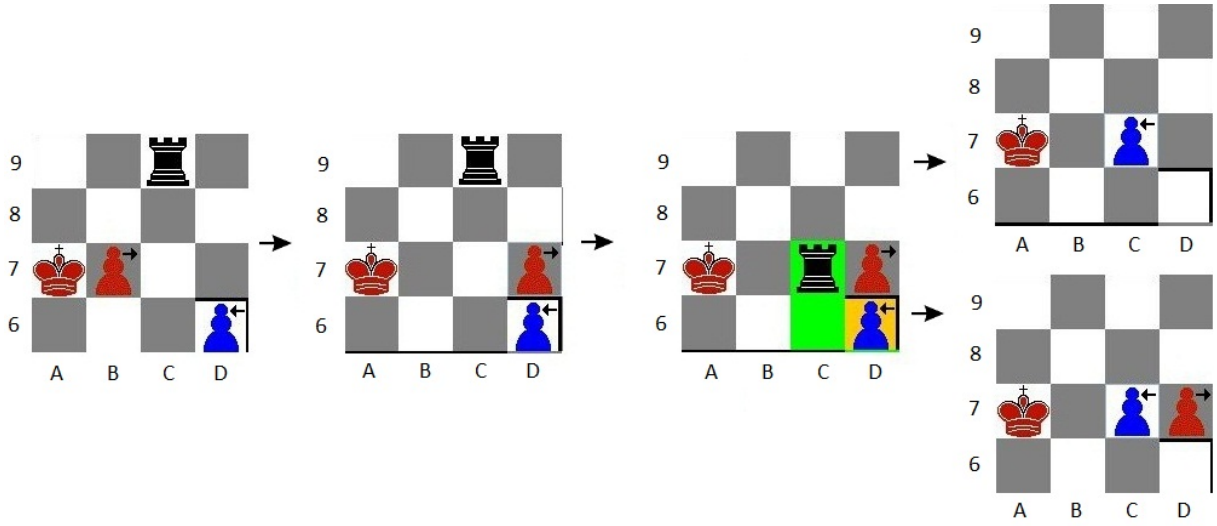


Figure 2.6: When the Pawn makes an en-passant move and there is a piece on the destination square, the Pawn can decide whether it only captures the piece on the destination square or also the Pawn.

2.2.4 En-Passant

The en-passant move can be made against any opponent player. Figure 2.6 shows a special case of the en-passant move in multi-player Chess. The red Pawn ♟ on B7 moves two squares ahead. The black player is on turn and moves the Rook ♜ from C9 to C7 which has been passed by the Pawns ♟ double step. The blue player can now capture the black Rook ♜ on C7. If the Rook ♜ would not have been moved to that square the blue Pawn ♟ could make an en-passant move against the red Pawn ♟. Because of this, Blue can decide if the capturing move against the Rook ♜ is a normal capturing move or also an en-passant move. If he decides that it is an en-passant move then he captures additionally the red Pawn ♟ on D7. Therefore it is possible to capture two opponents' pieces with one move.

2.2.5 Elimination

When a player is defeated, all of his pieces are removed from the board and the remaining players continue play. The winner gets one point. If the game is drawn all remaining players share one point, e.g., if two players are left both get half a point.

2.2.6 Hanging King

In addition to a checkmate, there exists another way to defeat an opponent. In multi-player Chess, the player on turn can be able to capture an opponent's King without setting the King in check. This occurs when another opponent moved an interposed piece away. This is illustrated in Figure 2.7 and is called Hanging King. In the example it is White's turn. If White moves his Bishop ♗, the red Queen ♛ can directly attack the black King ♚. Black is defeated directly after White's turn. The red Queen ♛ does not have to move to the Kings square to defeat Black.

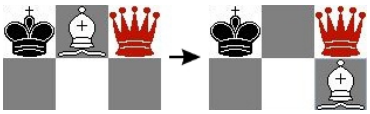


Figure 2.7: Hanging King

2.2.7 Passing

If there are more than two players left and the player on turn has no legal move he forfeits his turn. If there are only two players left and the player on turn has no legal move, the game is drawn.

2.3 Complexity

This section determines the complexity of multi-player Chess. First, the state-space complexity is calculated in Subsection 2.3.1. The game-tree complexity is computed in Subsection 2.3.2 and finally a comparison to other games is given in Subsection 2.3.3.

2.3.1 State-Space Complexity

The number of legal game positions reachable from the initial position is called state-space complexity (Allis, 1994). It is often hard to calculate this number and for simplicity an upper bound is computed. Shannon (1950) used the formula

$$\frac{64!}{32!(8!)^2(2!)^6} = 4.6347267 \times 10^{42}$$

to estimate the number of possible positions in regular Chess. The numerator is the number of squares on the board (8×8). The first number of the denominator, 32, is the number of empty squares if all pieces are placed on the board. For each player there are 8 Pawns on the board. Switching two Pawns of the same colour do not change the board and so the number of possible position has to be divided by 8! for each player. The same counts for the Bishop, the Rook and the Knight. Therefore the number of possible positions has to be divided by $2!^6$. Adapting this formula to multi-player Chess results in

$$\frac{160!}{96!(8!)^4(2!)^{12}} = 4.39181006 \times 10^{112}.$$

In comparison to regular Chess the size of the board is bigger ($14 \times 14 - 4 \times 3 \times 3 = 160$) and there are more empty squares on the board ($160 - 4 \times 16 = 96$). This shows that the state-space complexity of multi-player Chess is approximately 10^{70} times more complex than of regular Chess.

2.3.2 Game-Tree Complexity

The average branching factor to the power of the average game length is called game-tree complexity (Allis, 1994). To obtain the average branching factor and the average game length, 200 games with the *paranoid*, *max^n* and the *best-reply* algorithms were simulated with a thinking time of 5 seconds for each move. The 200 games contains games where only one of the three algorithms were used, games where two algorithms were used and also games were all three algorithms played against each other. The average branching factor of these 200 games is 37.3 and the average game length is 558.34. Therefore the game-tree complexity of multi-player Chess is $37.3^{558.34} = 3.5 \times 10^{878}$.

2.3.3 Comparison to Other Games

In this subsection, the state-space complexity calculated in Subsection 2.3.1 and the game-tree complexity computed in Subsection 2.3.2 are compared to other games. The information about the other games are taken from Van den Herik, Uiterwijk, and Van Rijswijck (2002). Figure 2.8 illustrates that only Go (19×19) of the chosen games has a larger state-space complexity than multi-player Chess. Moreover, multi-player Chess has the largest game-tree complexity. In conclusion, multi-player Chess has a quite high complexity.

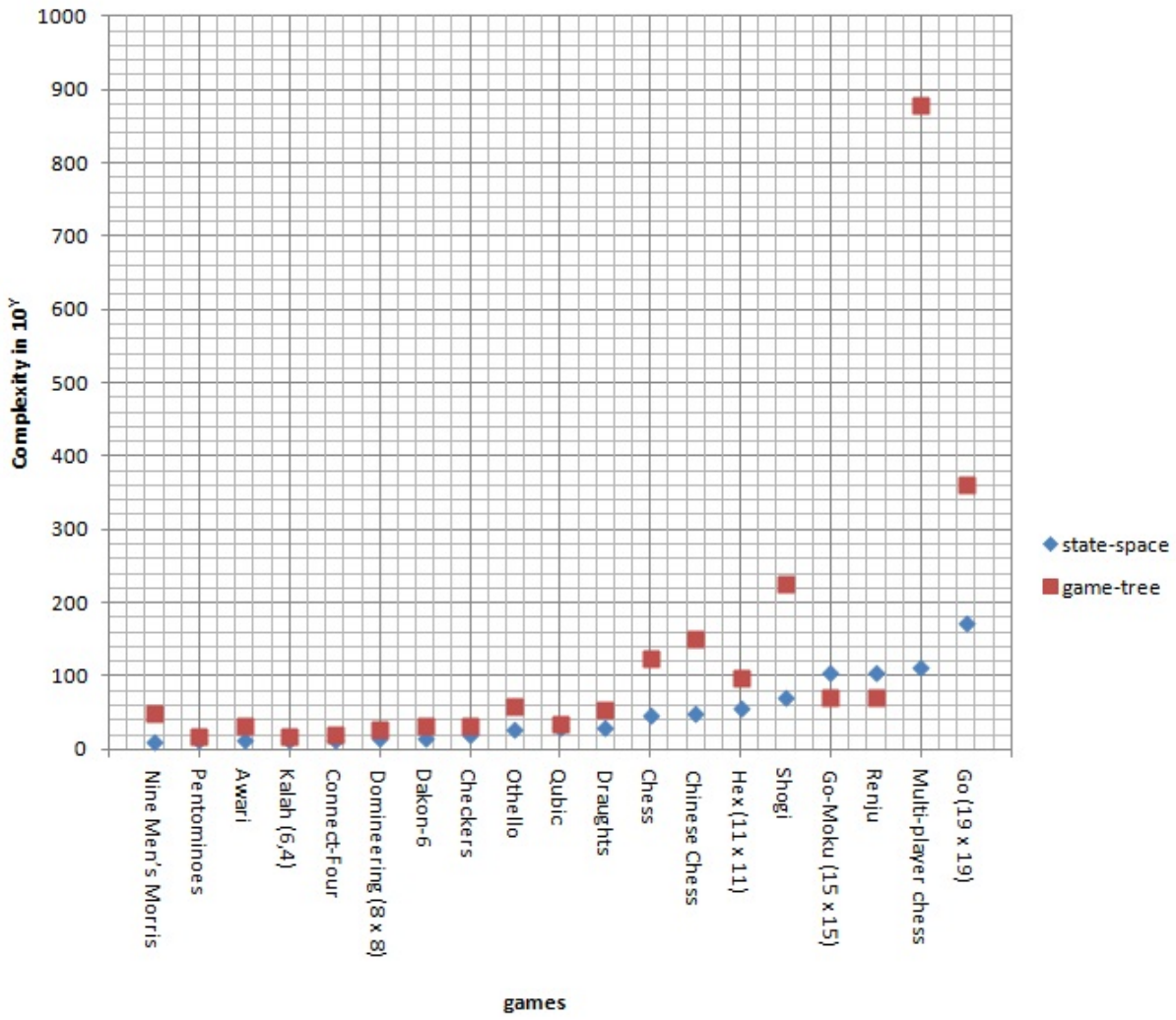


Figure 2.8: The state-space and game-tree complexity of different games

Chapter 3

Search Techniques

As explained in Subsection 2.3.1, multi-player Chess is quite complex and therefore it is not possible to assign a decision to each possible board position. An evaluation function (described in Chapter 5) is regarded to assign a value to each leaf node in the game tree (see Section 1.2). Moreover, a decision rule is required which defines how the values of the leaf nodes are propagated up the game tree. The evaluation function and the decision rule dictate the strategy of a player (Sturtevant, 2003a). Evaluation functions are domain dependent which means that an evaluation function for e.g., Chess cannot be reused for other games. But the generic decision rules can be used for many games. First in this chapter, a decision rule called *minimax* for two-player games is explained in Section 3.1. Enhancements of this decision rule are explained in Section 3.2. Decision rules for multi-player games are explained in Section 3.3 and quiescence search is described in Section 3.4.

3.1 Two-Player Search Algorithms

In the following, the most popular algorithm for two-player games, called *minimax*, is described. Further, $\alpha\beta$-pruning is explained.

3.1.1 Minimax

The powerful *minimax* algorithm (von Neumann, 1928) can be applied to two-player games in which both players play against each other. The player on turn is called max-player because he tries to maximize his value and by this his chance to win. The other player, called min-player, tries to minimize the max-player's value which results in maximizing his own chance to win. This simple but successful decision rule is illustrated in Figure 3.1. There the player on turn has two possible moves (Move *m1* and Move *m2*) and his opponent has two possible moves for each of the current player's moves (Moves *m3*-*m6*). The max-player has to make the decision which move he considers best. This player knows that the opponent wants to minimize the value. Therefore he expects the min-player to play Move *m4* after Move *m1* and Move *m6* after Move *m2*. The max-player can decide between the value 1 and 3, and chooses to play Move *m2*. Thus, 3 is the minimax value of this simple game tree. The values of the leaf nodes are normally determined by an evaluation function which is explained in Chapter 5 for multi-player Chess.

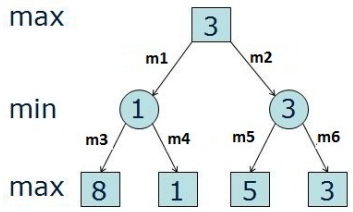


Figure 3.1: Simple minimax tree with the minimax value of the nodes in the corresponding boxes

The values of the other nodes are propagated up the tree as explained above. The pseudo-code for this algorithm can be seen in Algorithm 3.1.1. Usually, it is not possible to search the complete game tree and therefore a parameter defines the search depth of the algorithm.

Algorithm 3.1.1 Pseudo-code for the minimax-algorithm

```
1:  function MINIMAX(node, depth)
2:      if (depth == 0 or node is terminal)
3:          return evaluate(node)
4:
5:      if (node is max-node)
6:          maxValue = −∞
7:          ForEach (child of node)
8:              maxValue = max(maxValue, minimax(child, depth-1))
9:          return maxValue
10:
11:     if (node is min-node)
12:         minValue = ∞
13:         ForEach (child of node)
14:             minValue = min(minValue, minimax(child, depth-1))
15:         return minValue
16:
17: end function
```

$\alpha\beta$-Pruning

The game tree usually grows exponentially in depth and therefore it is not possible to reach a high search depth with the simple *minimax* algorithm. But the deeper the search goes, the more accurate the calculated value is. The *minimax* algorithm can be enhanced by cutting off branches of the tree without losing information. This technique is called pruning. Pruning is quite important for the *minimax* algorithm because it is possible to reach a higher search depth by reducing the size of the tree. The most successful pruning technique for the *minimax* algorithm is called $\alpha\beta$-pruning (Knuth and Moore, 1975). "It is based on a window of values between which a player's score is guaranteed to fall" (Sturtevant, 2003a). The lower value, called α, is the minimal minimax value of the tree and the upper value, called β, is the maximal minimax value of the tree. At the beginning of the search α is set to -∞ and β is set to ∞. During the search the max-player updates the lower bound of the nodes and the min-player updates the upper bound. If the bounds are changed in a way that α is greater or equal then β, the corresponding subtree can be pruned because the players would never choose to play the moves which lead to this subtree. The pseudo-code of the algorithm can be found in Appendix A.1.1.
Figure 3.2 illustrates the $\alpha\beta$-pruning with a search depth of 3. *Minimax* is a depth-first search. First, the two lower left nodes are evaluated. In this case, the max-player at Node *(c)* can choose between the values 3 and 4. As he wants to maximize the value, he chooses to play Move *m7*. Therefore, the min-player knows at Node *(b)* that the value is 4 if he chooses to play Move *m3*. The min-player tries to minimize the value and so it is known that the value of the corresponding node never becomes higher than 4. The evaluation function assigns the value 8 to the Node *(g)* and therefore the value at Node *(d)* will be at least 8. However, it is known that the min player at Node *(b)* chooses a move which at most gives the value 4. This value is used as β-value at Node *(d)*. At Node *(d)*, α is 8 and β is 4. As $\alpha \geq \beta$ holds, the other three nodes do not have to be investigated because no matter which value they get, the players will never choose a move combination which would lead to these nodes. After searching the left subtree, the max-player at Node *(a)* knows that he will at least achieve the value 4. The α value of Node *(a)* becomes 4 and the window of possible values is now (4, ∞). After investigating the left subtree of Node *(l)*, the node can at most get the value 2. Because it is known that the value of Node *(a)* is at least 4, the max-player at the root node will never choose to play Move *m2*. Thus, the complete right subtree of Node *(l)* can be pruned.
This example of a simple tree shows the importance of the $\alpha\beta$-framework. Considering a game with an average branching factor larger than 37 as explained in Subsection 2.3.2, the framework saves much

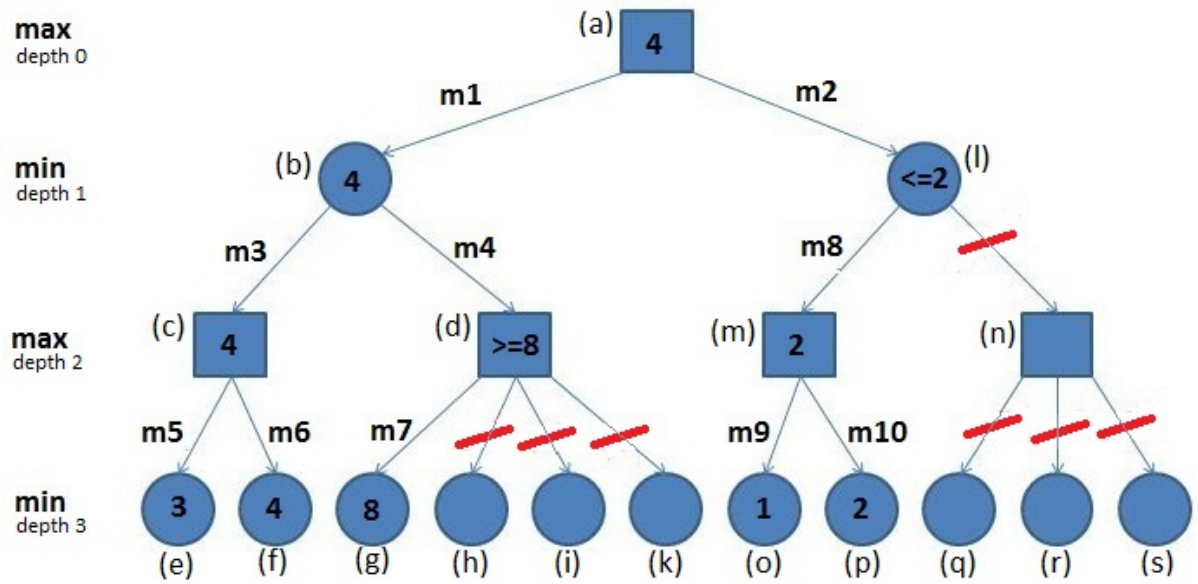


Figure 3.2: Minimax tree with $\alpha\beta$-pruning

computation time by simply passing around the two values α and β. The simple *minimax* algorithm investigates b^d nodes, where b is the average branching factor and d the search depth. Using $\alpha\beta$-pruning, the number of nodes can be reduced to $b^{d/2}$ in the best case (Knuth and Moore, 1975).

3.2 Search Enhancements for Improving Performance

The number of nodes pruned by the $\alpha\beta$-framework can be increased by using some additional techniques. In this section, the techniques used in the program are reported.

3.2.1 Move Ordering

As explained in Subsection 3.1.1, $\alpha\beta$-pruning has a best case of $b^{d/2}$. Move ordering is a good way to push the $\alpha\beta$-framework towards the best case. Figure 3.3 reveals this behaviour. In the left tree no pruning is possible. If the moves are ordered optimally, as seen in the right tree, many cutoffs can be performed.
There are two ways to order the moves, static move ordering and dynamic move ordering. Domain knowledge is required for static move ordering. In the program, the static move ordering orders the moves in such a way that capturing moves are considered first and non-capturing moves afterwards. The capturing moves themselves are ordered regarding the value of the captured piece (see Subsection 5.1.1). Additionally to the static move ordering, three techniques for dynamic move ordering are used in the program. Transposition tables are described in Subsection 3.2.2, the killer heuristic is explained in Subsection 3.2.3 and finally the history heuristic is described in Subsection 3.2.4.

3.2.2 Transposition Tables

A transposition table (Greenblatt, Eastlake, and Crocker, 1967) is an important technique to improve the performance of the $\alpha\beta$-framework. Most games are graphs instead of game trees. Thus, there are positions that can be reached by more than one path. These repeated positions are called transpositions. It is advantageous to detect these transpositions to avoid redundant searches (Breuker, 1998). A transposition table is a hash table that stores the results of previously performed searches. When the search encounters a position, it first checks whether the information about this position is stored in the transposition table. In the program, the following information about a position is stored in the table.

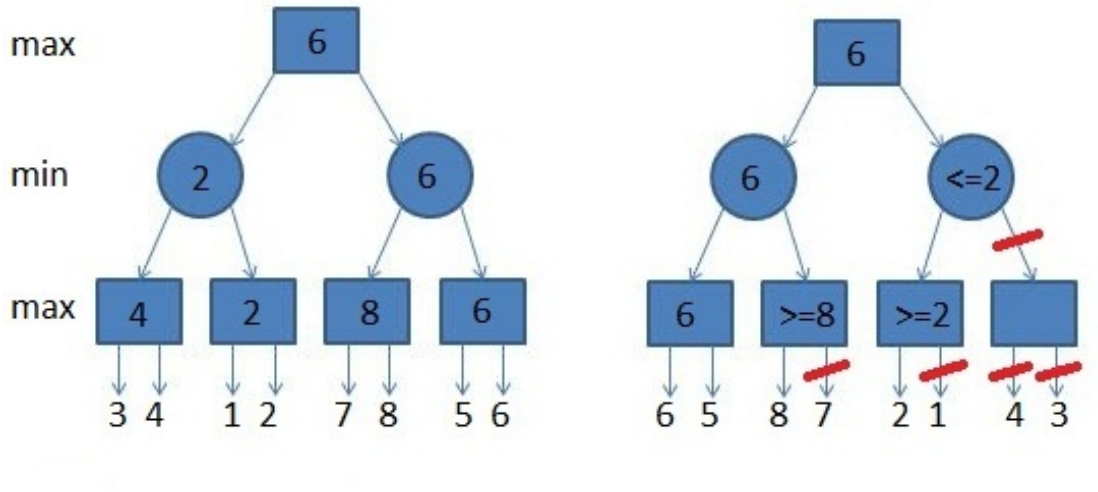


Figure 3.3: Improved pruning by move ordering

- **The search depth of the subtree**
  The search depth of the subtree is regarded to decide how the information from the table can be used. If the search depth is equal or larger than the depth the program wants to perform at this position, the best move and the corresponding values can simply be returned without any further search at this subtree (if the value is the real value and no upper or lower bound). If the depth of the table entry is not sufficient, or does not contain the right bound for a cutoff, the stored move can be put in the first position of the move ordering.

- **The best move found**
  The best move is the move which the player on turn considers best or the move which leads to a cut-off. If the search depth was sufficient, the search algorithm can simply return the best move and the corresponding values. In that case the move generator does not have to create all possible moves which saves a lot of time.

- **The values of best move found**
  The values the evaluation function (see Chapter 5) and decision rule have assigned to each player have also to be stored.

- **The flag**
  The flag defines if the stored values are exact values of the search or just lower or upper bounds which led to a cutoff. If the values are not exact values and they do not lead to a cutoff in the current search, then the subtree has to be searched again even if the search depth was sufficiently large.

- **The move counter**
  The move counter is required for the replacement scheme explained below. It stores the number of moves made so far in the game.

- **Hash key**
  To store the board in the transposition table first a unique hash value of it has to be created. Then a part of it is used as an index of the table. Some different board position has the same index but a different hash value. Therefore this value has to be stored to be sure that the board position found in the table is really the same board position as the current one.

The number of positions searched in a game exceeds the memory constraints of the system it runs on. So not all positions can be stored. One transposition in the program needs 80 bytes memory space. Each player has its own transposition table. If each of the four players store one transposition in the table,

320 Bytes memory is required. In the program each player is allowed to store $2^{19} = 524288$ entries in his table. Therefore, the maximum space for the transposition tables is roughly 170 MB. To store the values, the board position has to be transformed into a hash value, in the program a value of 64 bits. The last 19 bits are used as the index of the hash table. Zobrist hashing (Zobrist, 1970) is used to calculate this value. With Zobrist hashing it is possible to update the hash value incrementally which makes it fast.
If a new position is reached in the game and the last 19 bits of the hash value of this board is already used as index in the transposition table, then a replacement strategy is needed to decide which of the board position should be stored in the table. Therefore the move counter is stored in the transposition table. Even if the board stored in the transposition table has a larger search depth than the current board position, the current board position can replace the old one when the old position occurred in a previous search. In this thesis, the new value is stored if the length of the path leading to the position plus the depth of the subtree is bigger or equal than the corresponding value of the stored position.
There exists one problem in the application of transposition tables for some games - also regular Chess and multi-player Chess - called Graph-History-Interaction Problem (Campbell, 1985) which was solved by Kishimoto and Müller (2004). This problem occurs when the position is evaluated differently when reached via different paths. In multi-player Chess, there is a problem regarding the rule explained in Subsection 2.1.5 that the game is over if a board has repeated for the third time. The transposition table is in general not able to detect how often a position occurred and so a player in a winning position could play the best move stored in the transposition table even if this move would lead to a draw. In the program the problem is tackled by also using the number of same board positions so far to calculate the hash key of the position.

3.2.3 Killer Heuristic

The killer heuristic (Akl and Newborn, 1977) is a dynamic move ordering technique that improves the efficiency of $\alpha\beta$-pruning. Killer moves rely on the assumption that most moves do not change the board position too much. The killer heuristic tries to produce an early cutoff by first considering a legal move that led to a cutoff in another branch of the tree at the same depth. In the program, three moves per ply are stored as killer moves. If a non-killer move leads to a cutoff, it replaces the oldest killer move of that ply and becomes the killer move considered first at the next search of that ply. It is good to have more than one killer move per ply because it prevents from forgetting a good killer move to early.

3.2.4 History Heuristic

The dynamic move ordering technique called history heuristic was invented by Schaeffer (1983). At the beginning of a game, a table of all possible moves in the game has to be created. In multi-player Chess, there are 160 squares on the board and therefore a table of size 160×160 is required to store one value for each possible move. The piece type making the move is not considered. All values of the table are set to 0 at the beginning. At every internal node, the table entry for the best move found at that node gets incremented by a value. The added value can freely be chosen but typically it depends on the depth of the subtree because the deeper the search depth is, the more accurate the best move is. Two popular approaches are to use 2^{depth} or $depth^2$. In the program the last version is used and the values from the table are used to order the non-capturing moves. The values in the table can be maintained over the whole game. Each time a player has to make a new move the scores in the table are decremented by a factor. In the program the values are divided by 10. The history heuristic has some disadvantages. One of them is that the history heuristic is biased towards moves that occur more often in a game than others (Hartmann, 1988). To overcome this disadvantage, a relative history heuristic can be applied (Winands *et al.*, 2006).

3.2.5 Iterative Deepening

Iterative deepening (De Groot, 1965) breaks the search process into multiple iterations. Each iteration searches to a greater depth than the previous iteration (Russell and Norvig, 2010). Iterative deepening is used as a time management strategy. Each player in the game gets an amount of time for finding a move instead of a searching to a fixed depth. This is quite important for comparing the strength of different decision rules like max^n (see Subsection 3.3.1) and *paranoid* (Subsection 3.3.2). It was shown that this

technique is also beneficial in the $\alpha\beta$-framework because the result of the last iteration can be used for the dynamic move ordering in the new iteration which often leads to an early cutoff. When the time for searching is over, the algorithm stops the search at the current depth. In this thesis, the best move from the previous search depth is played. In general, it is also possible to use the information gained from the partial search at the current depth. The best move from the previous depth is searched first and thus, the value of this move is known first. When another move results in a higher value, it can replace the best move and be played when the time for searching is over.

3.3 Multi-Player Search Algorithms

In this section the decision rules for deterministic multi-player games with perfect information are described. First *max*^{n} is described in Subsection 3.3.1, then *paranoid* is described in Subsection 3.3.2 and finally *best-reply search* is explained in Subsection 3.3.3.

3.3.1 Max^n

Max^n (Luckhart and Irani, 1986) is the generalization of the *minimax* algorithm for multi-player games with n players (Sturtevant, 2003a). The root player assumes that each player tries to maximize his own score. The leaf nodes of the tree have n instead of just one value, where the i^{th} value represents the score of the i^{th} player. The pseudo-code can be seen in Algorithm 3.3.1.

Algorithm 3.3.1 Pseudo-code for the max^n-algorithm

```
1:  function MAXN(node, depth)
2:      if (depth == 0 or node is terminal)
3:          return evaluate(node)
4:      maxValues = maxN(firstChild, depth-1)
5:      ForEach (child of node)
6:          values = maxN(child, depth-1)
7:          if (values[playerOnTurn] > maxValues[playerOnTurn])
8:              maxValues = values
9:
10:     return maxValues
11: end function
```

Figure 3.4 illustrates a tree in a four-player game. At Node *(d)* the third player is on turn and has the choice between the values (1,2,3,4) and (4,3,2,1). As this player wants to maximize his own value, he chooses the first values and gets the value 3 instead of 2. When this decision rule is applied to the whole tree, the values of this game tree are (2,3,3,2). In the program a paranoid tie-breaking rule is used. When there exists one node in the tree where a player has the choice between two moves equally good for him, he selects the move that minimizes the value of the root player.

Shallow Pruning

One of the disadvantages of *max*^{n} is that only shallow pruning is possible without speculation (Sturtevant, 2003a). Shallow pruning is illustrated at Node *(o)* in Figure 3.4. After searching the left subtree of Node *(c)*, the second player knows that he will at least get the value 4 at Node *(c)*. After searching Node *(n)* the third player knows that he will get at least the value 7 at Node *(g)*. In the given tree, the sum of all players is always 10. When Player 3 gets the value 7 at Node *(g)*, Player 2 will never play the move leading to this node, because he knows that the maximum value he can reach at that node is $10 - 7 = 3$. He knows that by playing the move leading to Node *(f)* he will get the value 4 which is bigger than the maximum value he can reach at Node *(g)* and so Node *(o)* can be pruned. Shallow pruning is not as powerful as $\alpha\beta$-pruning. In the best case, it converges to $b^{0.5d}$ as b becomes large. But in the average case, no asymptotic gain can be expected (Korf, 1991). The main problem of this pruning is that only the values of 2 out of n players gets compared and thus it is unlikely that the sum of 2 players exceed the maximum sum available in this game. Therefore the lookahead is limited compared to the *paranoid* algorithm. The pseudo-code of the *max*^{n} algorithm with shallow pruning can be found in Appendix A.2.1.

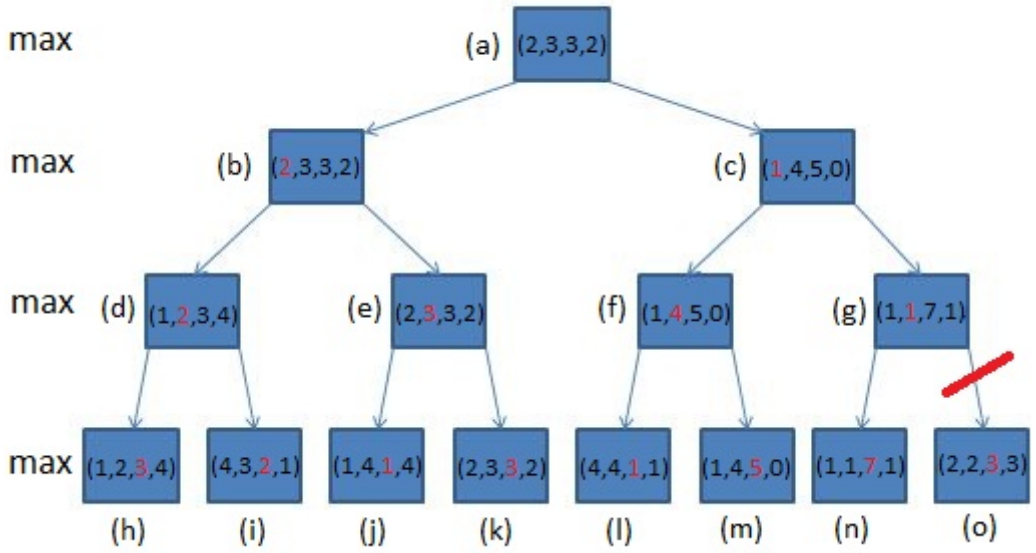


Figure 3.4: The max^n decision rule applied to a tree

3.3.2 Paranoid

The *paranoid* decision rule (Sturtevant and Korf, 2000) reduces the game to a two-player game and thus the $\alpha\beta$-framework can be applied. The root player assumes that all the opponents build a coalition against him. The root player is the only max-player and all the other players are min-players which try to minimize the value of the root player. This is depicted in Figure 3.5. There, all the players in the left subtree of Node *(a)* want to minimize the value and therefore the lowest value of this subtree is propagated to the root node. After investigating the left subtree, the max-player knows that he will at least get a value of 5. The first investigated leaf node of the right subtree has the value 2. The max-player will never play the move leading to this subtree and therefore all the other nodes of this subtree can be pruned. Beside this strength in pruning, the *paranoid* algorithm has one big weakness. Because of the unrealistic assumption that all opponent players form a coalition against the root player, suboptimal play can occur (Sturtevant and Korf, 2000). The pseudo-code of the *paranoid* algorithm is exactly the same as for the *minimax* algorithm (see Algorithm 3.1.1 and Algorithm A.1.1).

3.3.3 Best-Reply Search

Best-reply search (Schadd and Winands, 2011) allows only the opponent with the strongest counter move to play a move. The other opponents pass their move. Algorithm 3.3.2 presents the pseudo-code. The underlying assumption is that not all opponents try to minimize the value of the root player like the *paranoid* decision rule described in Subsection 3.3.2 does. The passing is one of the disadvantages of the algorithm because in some games it can lead to illegal positions. The advantage is that long-term planning can be done compared to the other multi-player decision rules. This can be seen in Figure 3.6. The max-player is again on turn after one opponents move and so more max-nodes are visited. In this figure, there are three min-players, each of them has two legal moves in each position. The prunings in the right subtree shows that $\alpha\beta$-pruning can be applied. Another advantage of this decision rule is that the playing style is less cautious compared to the *paranoid* decision rule (Schadd and Winands, 2011). At the same time, it has the disadvantage that opponent moves beneficial for the root player are not considered. The pseudo-code for the *best-reply search* with $\alpha\beta$-pruning is given in Algorithm A.3.1.

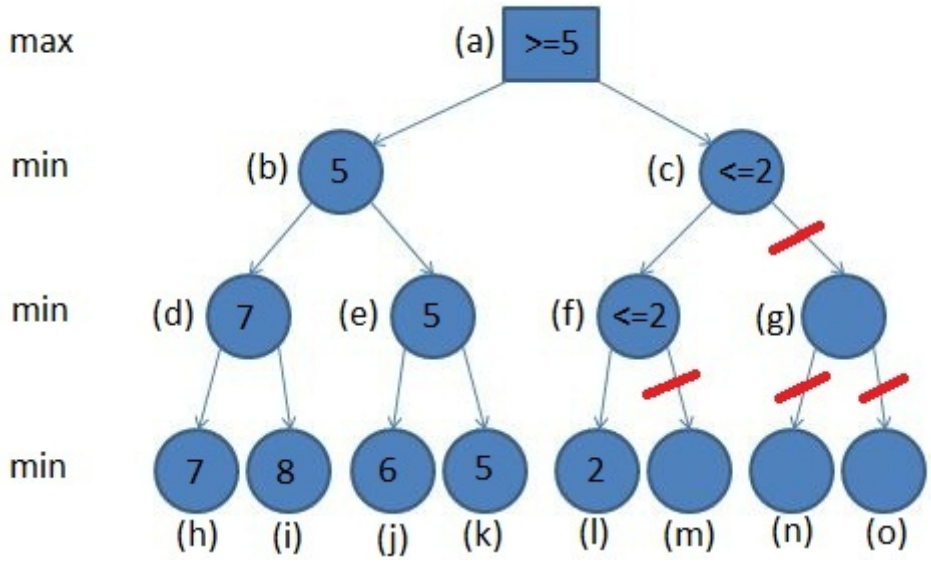


Figure 3.5: The paranoid decision rule applied to a tree

Algorithm 3.3.2 Pseudo-code for the best-reply search

```
1:  function BESTREPLY(node, depth, maxPlayer)
2:      if (depth == 0 or node is terminal)
3:          return evaluate(node)
4:
5:      if (maxPlayer)
6:          maxValue = −∞
7:          ForEach (child of node)
8:              maxValue = max(maxValue, bestReply(child, depth-1, false))
9:          return maxValue
10:
11:     if (not maxPlayer)
12:         minValue = ∞
13:         children = getChildrenOfAllMinPlayers(node)
14:         ForEach (children)
15:             minValue = min(minValue, bestReply(child, depth-1, true))
16:         return minValue
17:
18: end function
```

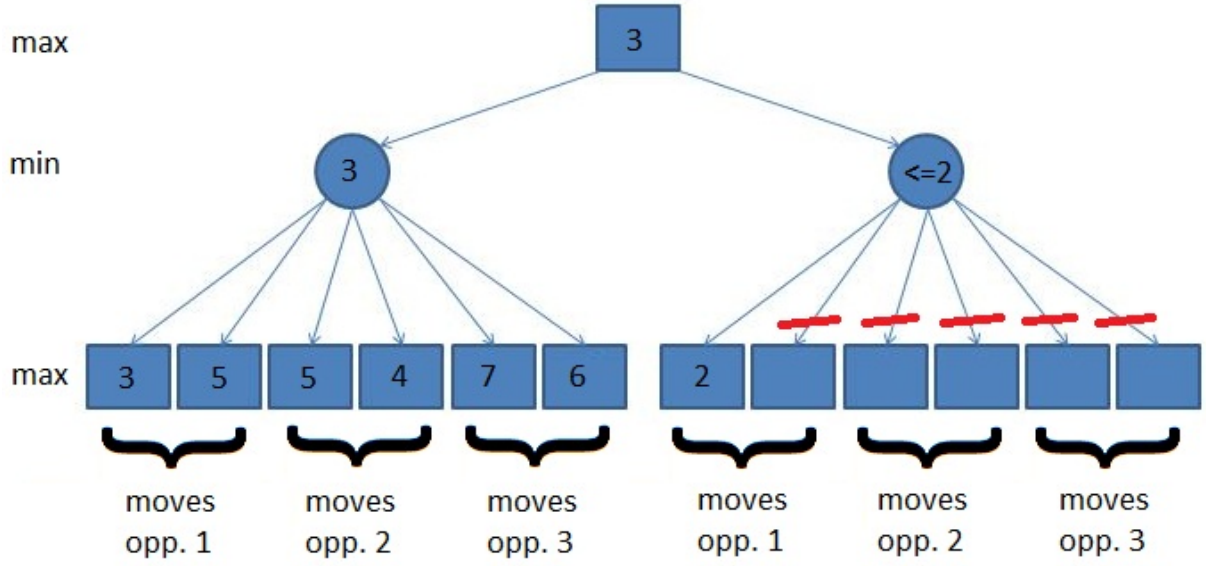


Figure 3.6: The best-reply decision rule applied to a tree

3.4 Quiescence Search

Quiescence search (Greenblatt *et al.*, 1967) is an additional search at a leaf node to dilute the horizon effect. Consider a Queen capturing a Pawn. If this move leads to a leaf node, the player thinks that it is a rather good move. But he does not consider if the opponent can capture his Queen with the next move. In that case, it is a bad move. The quiescence search is able to detect such bad moves by doing an additional search at the leaf nodes of the main search. In this additional search, fewer moves than in the main search are considered. In the program only capturing moves, promotion moves and checking moves are used in the quiescence search. The search terminates when the next player on turn does not have such a move. To terminate the quiescence search faster, checking moves are only considered in the first four plies. For more information about pruning techniques in the quiescence search and a more detailed explanation of quiescence search in general see Schrüfer (1989). The disadvantage of the quiescence search is that it cost additional computation time at the leaf nodes which can lead to a lower search depth in the given amount of time.
In the program, each kind of player uses the corresponding version of the quiescence search. The max^n search considers opponents capturing moves against all other players. The *paranoid* search considers only opponents capturing moves against the root player and the *best-reply* search does it similar, but considers all opponents as one player.

Chapter 4

Best-Reply Variants

In this chapter, first the weaknesses of *best-reply* are described in Section 4.1. Subsequently, ideas to overcome these weaknesses are presented in Section 4.2. The resulting search algorithms are described in Section 4.3. Section 4.4 gives an analysis of the complexity of the proposed algorithms and afterwards, the complexity of these algorithms in the domain of multi-player Chess is discussed in Section 4.4.6.

4.1 Weaknesses of the Best-Reply Search

Schadd and Winands (2011) mentioned two weaknesses of the *best-reply search*. First, passing can lead to illegal board positions, especially in card games. In the domain of multi-player Chess passing by choice is not allowed but passing does not lead to illegal board positions. If, for example, a player is in check and the *best-reply search* let this player pass, then it is no illegal board position because this position will be treated like a Hanging King as explained in Subsection 2.2.6. This is illustrated in Figure 4.1. White is the root player and moves his Queen ♕ to C9 as first step in the search, depicted in the left part of the figure. Both opponents' Kings are in check but only one is allowed to move in the *best-reply search*. In this case, the red King ♔ moves to A8. After that, White is on turn and so Black has a Hanging King and is defeated. Therefore, this is no illegal board position. Nevertheless, the illustrated position is another implicit weakness of the *best-reply search*. Passing leads to unreachable positions. The white player thinks he defeats one opponent by moving to square C9. But both opponents have to move out of check and therefore no player will be defeated. In such situations, the root player will probably do such moves although there might be better ones. During the experiments with the *best-reply search*, it was observed that the player often set an opponent in check when one of his pieces is threatened. Subfigure 4.2(a) depicts this. Black is on turn and moves his Knight ♞ from C10 to E9 and threatens the white Queen ♕. The white Queen could easily be saved by moving to another square. But White moves the Knight ♘ to C6 and threatens the red King ♔. In this position, he expected moving the red King ♔ out of check to be the strongest counter move and therefore his Queen ♕ to be safe. But after Red moves out of check, the black Knight ♞ will capture the white Queen ♕. Thus, the *best-reply search* will make a bad move. In summary, the first weakness of the *best-reply search* in the domain of multi-player Chess which has to be tackled is the emergency of unreachable board positions.
The second weakness of the *best-reply search* is that opponents' moves beneficial for the root-player are not considered. In the domain of multi-player Chess this is in particular important when one player

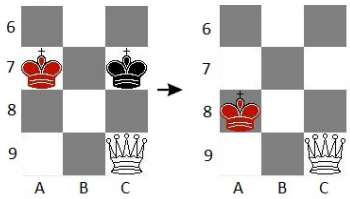
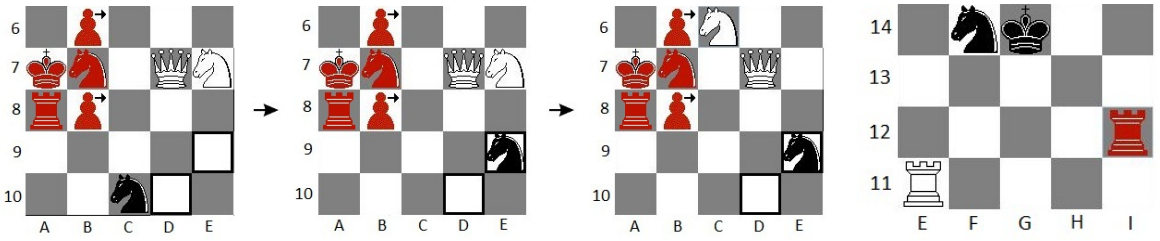


Figure 4.1: Hanging King in the best-reply search



(a) Attacking King instead of saving Queen

(b) The White best-reply search will not play a move which enables Red to checkmate Black

Figure 4.2: Weaknesses of the best-reply search

can be checkmated by a coalition of two opponents. One simple example of such a position is given in Subfigure 4.2(b) where White is to move. Black could be checkmated if the white Rook ♖ moves to E13 followed by the red Rook ♜ moving to I14. White will not consider Red to move the Rook ♜ to I14 and will probably not move his Rook ♖ to E13. *Max*^{n} and *paranoid* are able to detect such moves. But in the in the most cases, the *paranoid* player does not expect the opponent to do such moves because defeating one opponent will usually increase the root players value.

4.2 Ideas to Address the Weaknesses

The first weakness demonstrated in Section 4.1 can be eliminated by letting the passing opponents play a move. The first idea investigated in this thesis is to let these opponents play the best move ordering move (BMOM). This also solves the problem of illegal positions in other games. Two questions regarding the move ordering of those opponents arise:

1. *Is it better to use a paranoid or a max*^{n} *move ordering strategy?*
   To use a paranoid move ordering, which means the opponents will always capture a piece of the root player if possible, can be too pessimistic. The more realistic assumption that the opponents use a *max*^{n} move ordering strategy and therefore capture the most valuable piece, has also one disadvantage. If the first opponent can capture a second opponent's Rook, the root-player can capture a covered piece of the first opponent with his Bishop without expecting a recapturing. When the second opponent's Rook is covered and the own Bishop not, the first opponent will probably capture the Bishop. Therefore *max*^{n} move ordering can lead to bad moves. Both variants are tested in the experiments (see Subsection 6.4.1).

2. *Is it better to use only static or also dynamic move ordering techniques?*
   The static move ordering used in this thesis is based on capturing pieces. If no piece can be captured, the opponent explores the first move generated. There exist two ways to overcome this drawback. The first one is a better static move ordering (see Experiments in Subsection 6.4.2). The second one is to use dynamic move ordering, which can be smarter than a static move ordering but makes the search unstable.
   Applying the first technique for dynamic move ordering, transposition tables (see Subsection 3.2.2), is more complicated than in the other search algorithms. The transposition cannot be stored at each min-node. Consider the example that one opponent is searching and one is playing the BMOM. First, the first opponent searches the best move and the second opponent plays the BMOM. Storing that the BMOM is the best possible move in this position is simply wrong. When the first opponent plays the BMOM in the second phase of the search, the position will be detected in the transposition table and the BMOM would be considered as the best move without letting the second opponent search for it. Thus, the information about a position cannot be stored at each min-node. But if a position is stored in the transposition table, it can be used for the move ordering at each min-node. Using the killer move heuristic explained in Subsection 3.2.3 can also increase the strength of the *best-reply* variants. The advantage of using killer moves is that not only capturing moves but also

tactical moves can be considered for the BMOM. At the same time, this can be a disadvantage because tactical moves can be bad moves in another branch of the tree.
History heuristic, explained in Subsection 3.2.4, can also be able to gain a better move ordering. It can especially be useful in positions where one opponent has only non-capturing moves. History heuristics will prevent that always the first generated move is played as the BMOM in such a position.
All dynamic move ordering techniques are tested in the experiments (see Subsection 6.4.3) to find out which of them improves the proposed algorithms.

Let the non-searching opponents play a static move is only one idea. In this thesis, other variants, where all opponents play a move, are also investigated. These variants emerge from varying the number of opponents searching. Considering multi-player Chess with four players, when all three opponents search it is simply the *paranoid* algorithm. Let one opponent search is the proposed variant of the *best-reply search*. It can be a good idea to combine these both algorithms by let two opponents search. This idea is described in Subsection 4.3.2. Another approach, described in Subsection 4.3.3, is to let the third opponent pass instead of playing a move from the move ordering. The big advantage of the *best-reply search* is the long-term planning. Therefore, it is also interesting to see what happens if all opponents play a move from the static move ordering and only the root player is allowed to search (Schadd and Winands, 2011). This idea is described in Subsection 4.3.4. For all of the variants, the two questions mentioned above have to be answered by performing experiments.
The second weakness explained in Section 4.1 that opponents' moves beneficial for the root player are not considered, will not be resolved. Using a $\max^n$ move ordering for the opponents not searching may diminish it, but it will not solve the problem illustrated in Figure 4.2(b). This problem can be solved by considering checking moves in the static move ordering. The move generator used in this thesis does not detect whether moves are checking moves because of performance issues.

4.3 Best-Reply Variants

In this section, the proposed variants of the *best-reply search* are presented. The short form of the algorithm's name is given in the brackets in the title of the subsections. The indices indicate the number of the root player's competitors searching for the best move, followed by the number of the root player's competitors playing the BMOM, where C is the number of all root player's competitors. The short form is often used in Chapter 6.
In the program, for $BRS_{1,C-1}$ only search depths are considered for iterative deepening where all opponents has played the same number of moves. In the case of four players left in the game, search depths 2,3,6,7, etc. are skipped. Experiments had shown that this increases the strength of the algorithm. The other proposed algorithms perform better when no search depth is skipped.

4.3.1 One Opponent Searches ($BRS_{1,C-1}$)

The search for the strongest counter move combination of the opponents where only one opponent is allowed to search, can be divided into *n-1* phases, where n is the number of players in the game. In each of those phases, another opponent is allowed to search for the best move. Figure 4.3 illustrates a search for a four-player game. The three phases, after the root player's move, are as follows:

1. **First opponent searching**
   The first opponent searches the move he considers best and the other opponents play the best move ordering move (BMOM).

2. **Second opponent searching**
   The first opponent plays the BMOM. The second opponent searches the best move and the third opponent plays again the BMOM.

3. **Third opponent searching**
   The first opponent plays the BMOM as in the second phase of the search. The second opponent plays the BMOM and the third opponent searches for the best move.

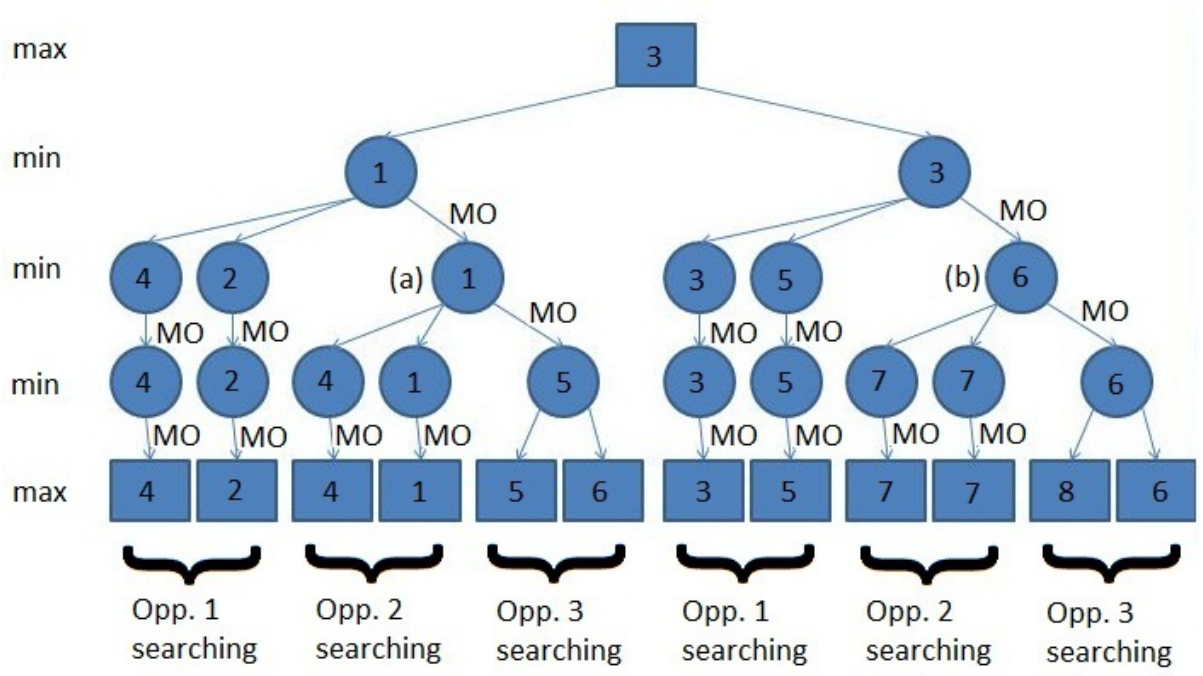


Figure 4.3: One opponent searching, two opponents play a move from the move ordering (MO)

The pseudo-code of the *best-reply search* where the non-searching opponents play the BMOM can be seen in Algorithm 4.3.1. This code is generic and so it can also be used for the other variants. For variants where the non-searching opponents pass, lines 24 and 36 have to be changed. The given code is a simplification of the real code. It builds a larger tree as illustrated in Figure 4.3 because in the code the *n-1* phases of the search are completely separated. Thus, in the case of four players, the move of the first opponent, leading to Node *(a)* and *(b)* in Figure 4.3, will be computed in the second and in the third phase of the search instead of only once.

4.3.2 Two Opponents Search ($BRS_{2,C-2}$)

The next algorithm proposed in this thesis let two opponents search and the other opponents play the BMOM. For four players, this is depicted in Figure 4.4 and can also be divided into three phases after the root player's move.

1. **First and second opponent searching**
   The first and the second opponent search for the best move, the third player plays the BMOM.

2. **First and third opponent searching**
   The first opponent searches for the best move as in the first phase of the search. The second opponent plays the BMOM and the third opponent searches for the best move.

3. **Second and third opponent searching**
   The first opponent plays the BMOM and the other opponents search for the best move.

Algorithm 4.3.1 is a simplification of the code which would build a larger tree than necessary because the *n-1* phases are completely separated.

Algorithm 4.3.1 Pseudo-code for $BRS_{1,C-1}$

```
1: function bestReplyWithStaticMoves(node, depth)
2:     if (depth == 0 or node is terminal)
3:         return evaluate(node)
4:
5:     if (node is max-node)
6:         searchPhase = null
7:         maxValue = −∞
8:         ForEach (child of node)
9:             maxValue = max(maxValue, bestReplyWithStaticMoves(child, depth-1))
10:        return maxValue
11:
12:    if (node is min-node)
13:        comment: first Node after maxNode
14:        if (searchPhase == null)
15:            minValue = ∞
16:            ForEach (phase of the search)
17:                searchPhase = currentPhase;
18:                if (isSearchMove)
19:                    comment: standard search
20:                    ForEach (child of node)
21:                        minValue = min(minValue, bestReplyWithStaticMoves(child, depth-1))
22:
23:                else
24:                    minValue = min(minValue, bestReplyWithStaticMoves(firstChild, depth-1))
25:
26:            return minValue
27:        else
28:            comment: searchPhase != null => second or third opponent
29:            minValue = ∞
30:            if (isSearchMove)
31:                comment: standard search
32:                ForEach (child of node)
33:                    minValue = min(minValue, bestReplyWithStaticMoves(child, depth-1))
34:
35:            else
36:                minValue = min(minValue, bestReplyWithStaticMoves(firstChild, depth-1))
37:            return minValue
38:
39:
40: end function
```

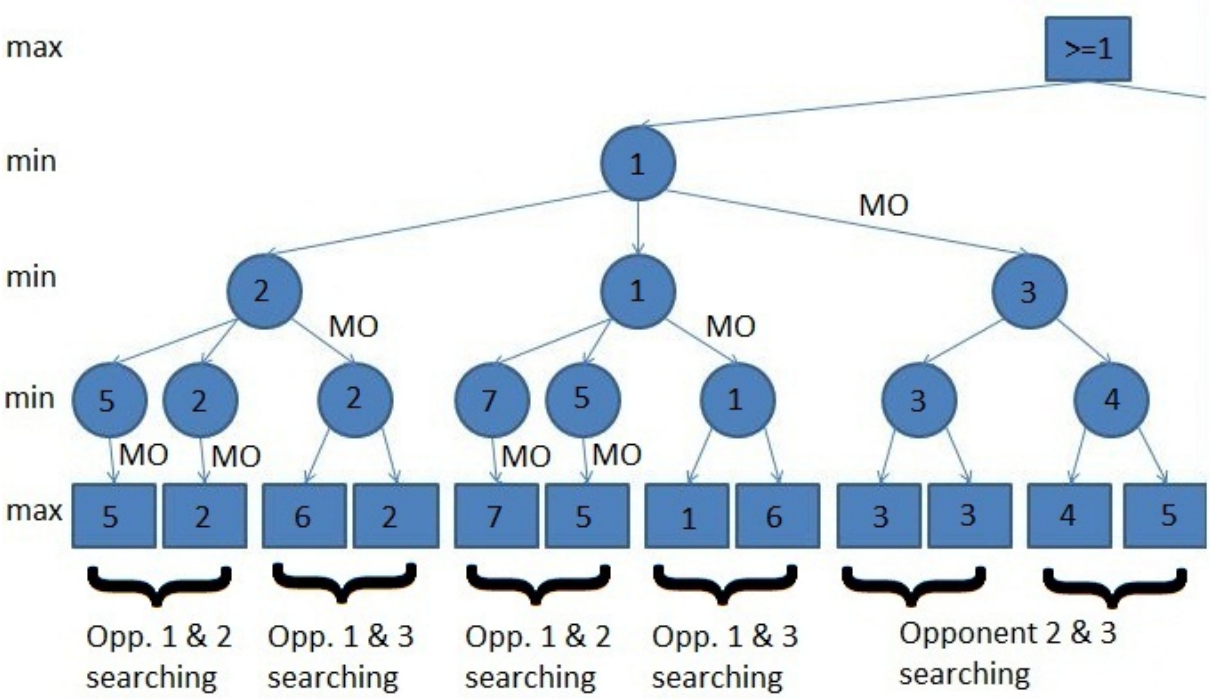


Figure 4.4: Two opponents searching, one opponent plays a move from the move ordering (MO)

4.3.3 Two Opponents Search, Remaining Opponents Pass ($BRS_{2,0}$)

This algorithm works similar as the algorithm proposed in Subsection 4.3.2. Again, two opponents search for the best move. The difference is, that the other opponents pass. Therefore the tree is smaller (see Figure 4.5). The disadvantages of this algorithm are the same as explained in Section 4.1.

4.3.4 No Opponent Search ($BRS_{0,C}$)

The last algorithm proposed in this thesis is illustrated for four players in Figure 4.6. In this search, all opponents play the BMOM. Therefore the tree is quite small which leads to a long-term planning of the root-player. Schadd and Winands (2011) already mentioned that it would be interesting to see how such an algorithm performs.

4.3.5 Importance of the Static Move Ordering

The static move ordering is already important for the *paranoid* and *max*^{n} algorithms. The better the move ordering is, the more efficient the pruning techniques are. For all the proposed *best-reply* variants except for $BRS_{2,0}$ (see Subsection 4.3.3), the move ordering is even more important. The BMOM is the root player's expectation of the move played by some or all of the opponents, depending on the *best-reply* variant. Therefore a bad static move ordering causes that the root player performs weak moves. Many experiments, described in Section 6.4, are performed to find a good move ordering.

4.4 Best-Case Analysis

In this section the best-case analysis of the algorithms is presented. First, the already analyzed best-case for *paranoid* and *best-reply* is stated in Subsection 4.4.1. The performed analysis of $BRS_{1,C-1}$ is described in Subsection 4.4.2 and the analysis of $BRS_{2,0}$ is given in Subsection 4.4.3. $BRS_{2,C-2}$ is analyzed in Subsection 4.4.4 and $BRS_{0,C}$ is analyzed in Subsection 4.4.5. In the following, b is the average branching factor of the tree, d is the search depth and n is the number of players.

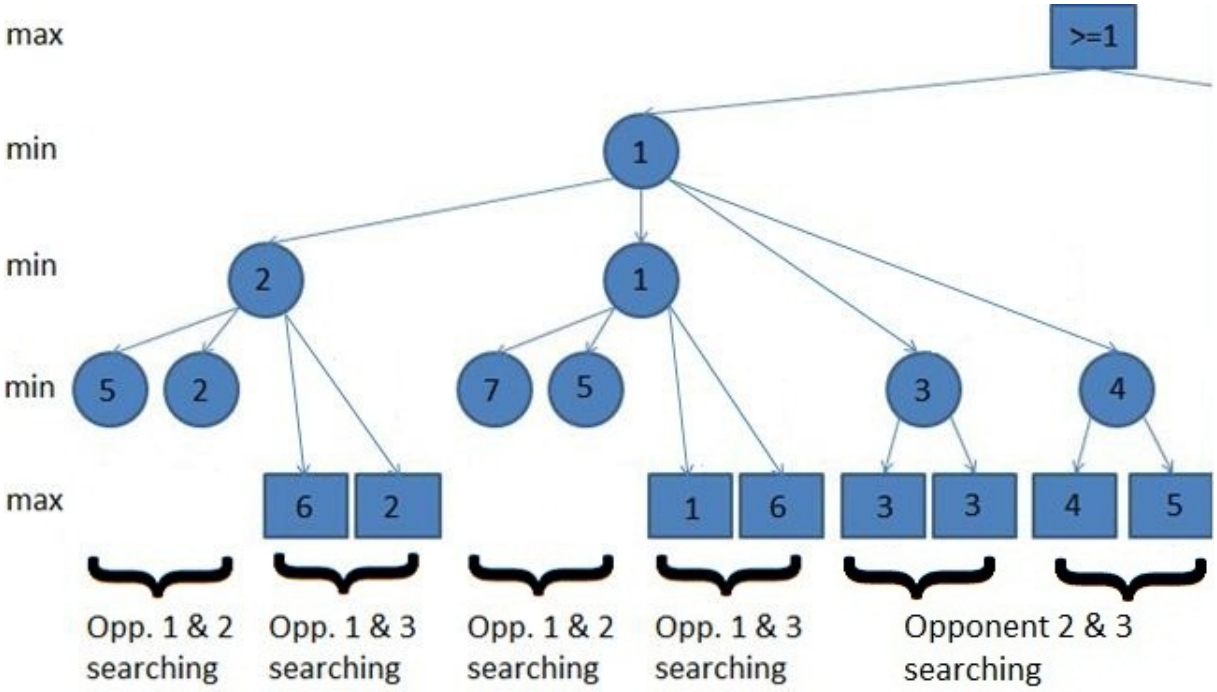


Figure 4.5: Two opponent searching, one opponent passes

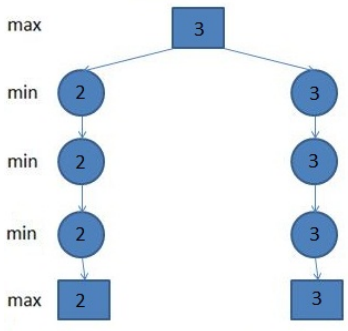


Figure 4.6: All opponents play the best move from the static move ordering

4.4.1 Paranoid and Best-Reply

Sturtevant (2003b) proofed the best-case performance of the *paranoid* algorithm as

$$\mathcal{O}(b^{d\times(n-1)/n}).$$

In this algorithm, the max-player knows his strategy and has therefore only to explore 1 node at a max-node. At each min-node, all b moves have to be investigated to find the best move. Therefore, the base of the formula is b. The exponent is $d \times \frac{n-1}{n}$ because for all of the $n-1$ opponents, a search has to be performed.
The best-case analysis of the *best-reply search*, done by Schadd and Winands (2011), results in the following:

$$\mathcal{O}((b \times (n-1))^{\lceil \frac{2\times d}{n} \rceil/2}).$$

The branching factor is $b \times (n-1)$ because the moves of all $n-1$ min-players has to be investigated at the min-layer. The search depth is reduced, because all min-players are considered as one player. Thus, the tree has two instead of n layers.
The explanation of the two formulas is simplified. The strategy for the min-player has also to be examined. But in both cases the complexity of the max-player's strategy exceeds to complexity of the min-player's strategy. A more detailed explanation of the formulas can be found in Sturtevant (2003b) respectively Schadd and Winands (2011).

4.4.2 $BRS_{1,C-1}$

The best-case analysis of $BRS_{1,C-1}$ (see Subsection 4.3.1) is accomplished for the minimal tree, depicted in Figure 4.7(a). The tree in the figure has a branching factor of four at each node.
Theorem:
The number of nodes explored by $BRS_{1,C-1}$ in the best-case is:

$$\mathcal{O}((n-1) \times (b + (n-2) \times (b-1))^{\lceil \frac{2\times d}{n} \rceil/2})$$

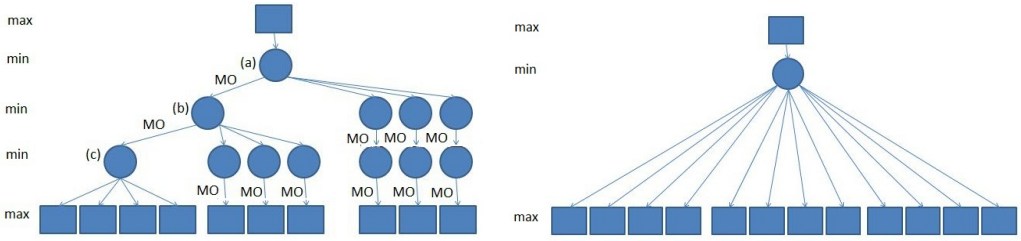
Proof:
The min-player is considered as a combination of the $n-1$ opponents. The search depth is reduced to $\lceil \frac{2\times d}{n} \rceil$ because the layers of n players are reduced to two layers. Analogous to the proof of Sturtevant (2003b) and Schadd and Winands (2011), a strategy is required to calculate the minimum number of nodes. The strategy for the max-player is as follows. This player knows his own strategy and has therefore only one move to search at a max-node. The number of explored leaf nodes for a min-player is $b+(n-2)\times(b-1)$. This can be seen in Figure 4.7(a). There are b leaf nodes in the most left subtree. For each of the other $n-2$ opponents, there are *b-1* leaf nodes. Calculating the strategy for the min-player is similar. The min-player knows his strategy and thus, only one move has to be searched at a min-node. At a max-node, all b children has to be searched. The total number of leaf nodes for the min-player and the max-player is

$$(b + (n-2) \times (b-1))^{\lceil \frac{2\times d}{n} \rceil/2} + b^{\lceil \frac{2\times d}{n} \rceil/2}.$$

Thus, in the best-case $BRS_{1,C-1}$ explores $\mathcal{O}((b + (n-2) \times (b-1))^{\lceil \frac{2\times d}{n} \rceil/2})$ leaf nodes.
There are fewer leaf nodes than in *best-reply*. But as Figure 4.7 depicts, there are more internal nodes. The number of additional internal nodes is larger than the number of saved leaf nodes. To indicate this computational overhead, the coefficient $n-1$ can be multiplied to the derived formula. The factor is $n-1$ because there are at most $n-2$ internal nodes per leaf node. Summing up the $n-2$ internal nodes and the leaf node itself leads to $n-2+1 = n-1$. Thus, the following formula is an estimation of the number of nodes in a $BRS_{1,C-1}$ tree.

$$\mathcal{O}((n-1) \times (b + (n-2) \times (b-1))^{\lceil \frac{2\times d}{n} \rceil/2})$$

For two players, the algorithm works identically to *paranoid*. Inserting n=2 to the formula leads to $\mathcal{O}(b^{\frac{d}{2}})$, which is also the best-case of a two-player *paranoid* search.



(a) Nodes in a $BRS_{1,C-1}$ tree (b) Nodes in a BRS tree

Figure 4.7: Comparison of a $BRS_{1,C-1}$ and a BRS tree

4.4.3 $BRS_{2,0}$

The best-case analysis for $BRS_{2,0}$, described in Subsection 4.3.3, is quite easy.
Theorem:
The number of nodes explored by $BRS_{2,0}$ in the best-case is:

$$\mathcal{O}(BRS_{2,0}(b,d,n)) = \begin{cases} \mathcal{O}((\frac{b^2\times(n-1)\times(n-2)}{2})^{\lceil\frac{2\times d}{n}\rceil/2}), & \text{if } n>3 \\ \mathcal{O}(b^{d\times(n-1)/n}), & \text{if } n\leq 3 \end{cases}$$

Proof:
For two and three players, the algorithm works exactly as *paranoid*. Therefore the best-case analysis from *paranoid* can be used for $n \leq 3$. To calculate the number of nodes for more than 3 players, all the min-players are considered as one player and therefore the exponent is $\lceil\frac{2\times d}{n}\rceil$. The strategy for the min-player is the same as for $BRS_{1,C-1}$. The min-player knows his strategy and has therefore only to examine one node at a min-node. At a max-node, b nodes have to be explored. Again, the strategy of the min-player is not important to calculate the complexity because the strategy of the max-player is much more complex. The max-player has to explore 1 node at a max-node. At a min-node, the calculation of the number of paths leading to leaf nodes is the well-known *urn problem* (Johnson and Kotz, 1977; Papula, 2001). 2 out of the $n-1$ opponents play a move, the other opponents pass. The order does not matter. For instance, Opponent 1 playing a move followed by Opponent 3 playing the move is considered as the same as Opponent 3 playing a move followed by Opponent 1 playing a move because only the first way is allowed in the proposed algorithm. Therefore,

$$\binom{n-1}{2} = \frac{(n-1)!}{2!\times(n-3)!} = \frac{(n-1)\times(n-2)}{2}$$

paths lead to leaf nodes. Each path leads to b^2 leaf nodes. There are much fewer internal nodes than leaf nodes. Thus, they can be neglected. In conclusion, the number of nodes in $BRS_{2,0}$ for more than 3 players are:

$$\mathcal{O}((\frac{b^2\times(n-1)\times(n-2)}{2})^{\lceil\frac{2\times d}{n}\rceil/2})$$

4.4.4 $BRS_{2,C-2}$

In this subsection, the best-case analysis of $BRS_{2,C-2}$, explained in Subsection 4.3.2, is given.
Theorem:
The number of nodes explored by $BRS_{2,C-2}$ in the best-case is:

$$\mathcal{O}(BRS_{2,C-2}(b,d,n)) = \begin{cases} \mathcal{O}((n-2)\times(\frac{b^2\times(n-1)\times(n-2)}{2})^{\lceil\frac{2\times d}{n}\rceil/2}), & \text{if } n>3 \\ \mathcal{O}(b^{d\times(n-1)/n}), & \text{if } n\leq 3 \end{cases}$$

Proof:
The formula to calculate the number of nodes for $BRS_{2,C-2}$ is similar to the formula for $BRS_{2,0}$. The number of leaf nodes are exactly the same. But $BRS_{2,C-2}$ has more internal nodes. To indicate this computational overhead, $n-2$ can be multiplied to the number of leaf nodes.

Table 4.1: Comparison of the complexity of different algorithms

d	n	b	$BRS_{0,C}$	BRS	$BRS_{1,C-1}$	$BRS_{2,0}$	$BRS_{2,C-2}$	Paranoid
3	3	37.3	37	75	147	1,391	1,391	1,391
4	3	37.3	227	644	1,263	15,532	15,532	15,532
5	3	37.3	1,391	5,565	10,834	173,392	173,392	173,392
6	3	37.3	1,391	5,565	10,834	1,935,688	1,935,688	1,935,688
7	3	37.3	8,497	48,067	92,945	21,609,326	21,609,326	21,609,326
8	3	37.3	51,895	415,161	797,377	241,238,769	241,238,769	241,238,769
9	3	37.3	51,895	415,161	797,377	2,693,103,168	2,693,103,168	2,693,103,168
4	4	37.3	37	112	330	4,174	8,347	51,895
5	4	37.3	228	1,184	3,456	269,654	539,309	783,264
6	4	37.3	228	1,184	3,456	269,654	539,309	11,821,967
7	4	37.3	1,391	12,522	36,234	17,421,191	34,842,382	178,431,435
8	4	37.3	1,391	12,522	36,234	17,421,191	34,842,382	2,693,103,168
9	4	37.3	8,497	132,457	379,852	1,125,504,656	2,251,009,312	40,647,572,534

4.4.5 $BRS_{0,C}$

The best-case analysis of the last proposed algorithm, $BRS_{0,C}$ (see Subsection 4.3.4), is given in the following.

Theorem:

The number of nodes explored by $BRS_{0,C}$ in the best-case is:

$$\mathcal{O}(b^{\lceil \frac{d}{n} \rceil})$$

Proof:

This algorithm reduces the search to a single-player game because no opponent performs a search. The exponent is $\lceil \frac{d}{n} \rceil$ because the opponents' layer are removed. In each layer, the player has the choice between the b moves.

4.4.6 Multi-Player Chess Analysis

Table 4.1 presents the estimated number of nodes in multi-player Chess for the given search depth d and number of players. The average branching factor b in multi-player Chess is 37.3 (see Subsection 2.3.2). The algorithms are ordered with regard to their long-term planning. $BRS_{0,C}$ needs to explore the fewest number of nodes to reach a given search depth, while *paranoid* needs the most. If, for instance, 4 players participate, $BRS_{1,C-1}$ needs to explore 379,852 nodes to have a lookahead of 3 own moves (search depth 9), while *paranoid* is only able to compute 4 depths completely with the same number of nodes and thus, has just a lookahead of 1 own move. The experiments in Chapter 6 will show whether the better long-term planning of the *best-reply* variants, especially of $BRS_{0,C}$ and $BRS_{1,C-1}$, is sufficient to outperform *paranoid*.

Chapter 5

Evaluation Function

In this chapter, the heuristic evaluation function is explained. First, the general features of the evaluation function are described in Section 5.1. Then, two features which are used in the opening are explained in Section 5.2. At the end of this chapter, the concept of "lazy evaluation" and how it is used in the program is explained in Section 5.3.

5.1 General Features

In most game programs an evaluation function is a linear combination of different evaluation features. The features of the evaluation function explained in this section are mainly based on the evaluation function of the two-player Chess program REBEL (Schröder, 2007). They are adapted in a way that they can be used for multi-player Chess. For an efficient implementation of an evaluation function in the domain of Chess, it is absolute necessary to use piece-square tables. These tables are used to store fixed values for each square. Thus, the value from this table can be used to evaluate a position instead of recalculating it each time. A good example for the piece-square table used in this thesis is the table for calculating the value of a Knight as explained in Subsection 5.1.3.
In the program, the values of all features for one player are summed up. In the end, the value for each player is divided by the sum of the values of all players. Therefore, each player has a value between 0 and 1, which indicates the probability to win. The values of all players sum up to 1.

5.1.1 Piece Value

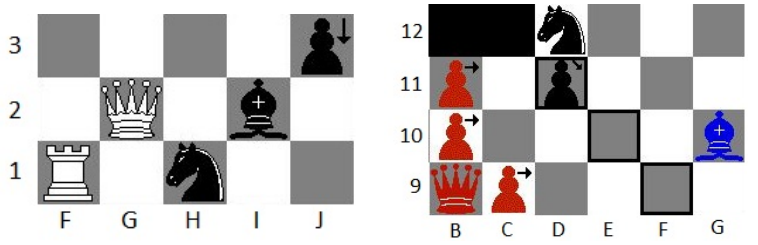
The most important feature of the evaluation function is the value of the pieces. The values used in the program are given in Table 5.1. In the most two-player Chess programs, the King has the value ∞ because when the King is defeated, the game is over. In multi-player Chess this value leads to a quite aggressive behaviour and thus to bad moves. The players try to defeat one opponent, without considering how many own pieces are captured. Some values for the King were tested. A value of 3 leads to reasonable play.

5.1.2 En Prise

The second most important feature of the evaluation function is called En Prise (French for "in danger"). En Prise are pieces which are defended insufficiently. For example in two-player Chess, a Rook which can be captured by an opponent's Bishop is not defended sufficiently even if it is covered. The Rook's value is higher than the Bishop's value (see Table 5.1) and therefore the opponent will probably capture

Table 5.1: The values of the pieces in multi-player chess

Queen ♕	Rook ♖	Bishop ♗	Knight ♘	King ♔	Pawn ♙
10	5	3	3	3	1



(a) An example for En Prise (b) The Bishop does not bring the Queen in a dangerous position

Figure 5.1: En Prise

Table 5.2: The relative value of a Knight

Number of possible moves	2	3	4	5	6	7	8
Relative value	2.82	2.85	2.88	2.91	2.94	2.97	3.00

the Rook. This is more complicated when the attacking piece has a higher value than the defending piece. Subfigure 5.1(a) illustrates such an example. The white Rook ♖ on F1 will capture the black Knight ♞ on H1. Subsequently, the black Bishop ♗ on I2 captures the white Rook ♖ . Then, the white Queen ♕ capture the black Bishop ♗ . White will start this capturing sequence because Black loses more valuable pieces. Therefore, the black Knight ♞ is in a dangerous position, called En Prise. The value of pieces which are in a dangerous position get decreased. In multi-player Chess, it is more complicated to use the En Prise feature. Given the position illustrated in Figure 5.1(a) with four players, White will not start the capturing sequence. The probability to win the game will decrease for both involved players because their pieces get captured while the other players do not lose a piece. In this thesis, the value of pieces from the root player get decreased by $\frac{5}{6}$ of its value if they are in a dangerous position (e.g, a Pawn in a dangerous position has the value $\frac{1}{6}$). Thus, the root player will avoid to move pieces into a dangerous position. The value of the opponents is decreased by the factor $\frac{1}{6}$. An opponent's Pawn in a dangerous position has the value $\frac{5}{6}$. If the factor would be higher, the root player will seldom capture an opponent's piece. If only one opponent's piece is able to attack a root player's piece and the root player is on turn before the capturing can happen, the value of the root player's piece is not decreased (see line 8 of Algorithm 5.1.1). In such a situation, the root player can move this piece before it gets captured. This prevents the root player to be even more pessimistic. Subfigure 5.1(b) illustrates such a position. Red is on turn and the search depth is 3. If the red Queen ♛ on B9 captures the black Pawn ♟ on D11, Red expects Blue to move the Bishop ♗ to E12 or F9 to bring the red Queen ♛ in a "dangerous" position, but the red Queen ♛ can easily escape. The last feature helps to detect such position as not dangerous. Therefore Red would not be too pessimistic and be able to capture the black Pawn ♟ . To evaluate whether a piece is in a dangerous position, a static exchange evaluator (SEE) (Reul, 2010) is used. The pseudo-code of the SEE — which returns the value by which the value of the given piece gets decreased — used in the program can be seen in Algorithm 5.1.1.

5.1.3 Knights Position

Knights are more powerful when they are placed near the centre of the board ("a Knight on the rim is grim"). Therefore, the value of the Knight is decreased if it is placed near the edge of the board. The number of possible moves if the board would be empty is used in the program to calculate the value. It would cost too much time to calculate this number each time the evaluation function is invoked and therefore the values of the Knight are stored in a piece-square table (see Section 5.1). The values used in the program can be found in Table 5.2.

Algorithm 5.1.1 Pseudo-code for the en prise feature

```
1: function getEnPriseValue(defendingPiece)
2:     returnValue = defendingPiece.getValue()
3:     if (defendingPiece is from root player)
4:         returnValue /= 6.0
5:     else
6:         returnValue /= 1.2
7:
8:     if (attackingPieces == 1 AND defending player is earlier on turn than attacker)
9:         return 0.0
10:    value = 0.0
11:    defenderValue = defendingPiece.getValue()
12:    attackerValue = 0.0
13:    while attackers left do
14:        if (value < 0.0)
15:     comment: Piece of defender can be captured, but defender already lost more valuable pieces
16:            return returnValue
17:        value -= defenderValue
18:        attackerValue = lowest value of remaining attackers
19:        if (value ≥ 0)
20:            comment: Defender is on turn, but attacker has already lost more valuable pieces
21:            return 0.0
22:
23:        if (defender left)
24:            value += attackerValue
25:        else
26:            return returnValue
27:        defenderValue = lowest value of remaining defenders
28:    end while
29:    return 0.0
30: end function
```

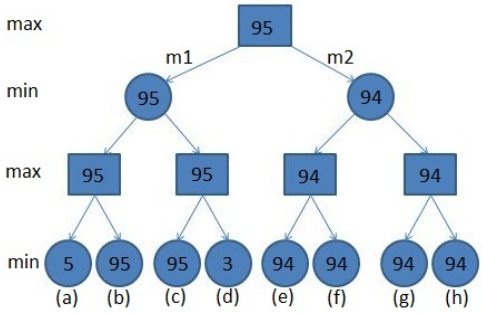


Figure 5.2: A random factor can change the decision of the root player

5.1.4 Random

A random factor is another feature used in the evaluation function. Beal and Smith (1994) performed experiments to show that it is beneficial to combine the existing evaluation function with a random factor. Figure 5.2 demonstrates that a random factor lets the player tend to play a move where the player has more promising options. In the given tree, without random the player will play the Move *m1*. It would probably be a good decision to play Move *m2* because all board position reachable after playing Move *m2* are good positions for the root player. A maximum random value of e.g., 10 can lead to play Move *m2* because it is probable that the value of the right tree gets more increased than the value of the left subtree. The value of the left subtree is $min(95 + b; 95 + c)$, while the value of the right subtree $min(max(94 + d; 94 + e); max(94 + f; 94 + g))$ is. The characters indicate the random value of the corresponding nodes (see Figure 5.2). If one of the Nodes *(b)* and *(c)* receive a low random value, the value of the left subtree is only increased by this low value. In the right subtree, the max-player can choose the node with the highest random number and therefore not every node needs to get a high random number to increase the value of the right subtree by a high number.
Calculating random numbers each time the evaluation function is invoked costs some time. In the program at the beginning of a game 100,000 random numbers are computed and stored in an array to save computation time.

5.1.5 King Safety

The sole objective of Chess is to capture the opponent's King. Thus, it is important to evaluate the pressure on the Kings. Figure 5.3 illustrates which squares are important to measure the pressure on the King. The important squares are marked with a ■, a ● or a ╳. The pressure on the King is measured in a number, called *counter*. This *counter* is initialised with the value 0. In the following, the steps for computing the king safety is explained for the white King ♔.

1. **Evaluating the King's position**
   To avoid that the King ♔ moves to the centre of the board or even to an opponent's side, a piece-square table is used. It is for instance good when the King ♔ is placed on a square from E1 to J1. The further the King ♔ is away from these squares, the more unsafe the King ♔ is. For a more detailed explanation of this square table and the king safety in general (for two-player Chess), see Schröder (2007).

2. **Measure the pressure on the ■ squares**
   For each of the opponents and for each kind of piece it have to be calculated how often the ■ squares can be reached by the opponent.

3. **Measure the pressure on the ╳ squares**
   For the ╳ squares, the same computation as for the ■ squares has to be executed. Additionally, the *counter* has to be incremented for each opponent who can reach one of the ╳ squares. If e.g., one opponent is able to reach two ■ squares by its next move and another opponent is able to reach one ■ squares, the *counter* gets increased by 3. If the attacked square is not covered by any of the white pieces, the *counter* gets incremented a second time.

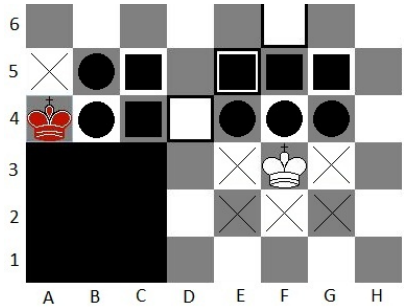


Figure 5.3: The squares around the King important for the King Safety

4. **Measure the pressure on the ● squares**
   The computation is similar to the calculation of the ⨯ squares. The only difference is that if the attacked ● square has no white piece on it, the *counter* gets additionally increased.

5. **Calculate the attack pattern**
   The result of step 2, 3 and 4 is a *counter* with a value indicating how much the pressure on the King is and the knowledge which of the opponents' pieces are able to attack the squares around the King. This knowledge is used to calculate a value for the attack pattern each opponent has. In Subtable 5.4(a), it can be seen which attack pattern is transformed to which value. This value for each opponent is added to the *counter*. The number of attackers is also an important factor. Three attacking opponents give more pressure on the King than only one attacker because they are able to play three consecutive moves against the defending player. Thus, the *counter* is increased by 2 if two opponents attack and by 5 if three opponents attack the King. The last step is to use the *counter* to decrease the value of the King. The higher the counter, the lower the King's value. If there is no pressure on the King the value is the default one. The value is decreased to less than the value of one Pawn if the pressure on the King is very high. The relative value of the King for the different values of *counter* can be seen in Subtable 5.4(b).

5.1.6 Mobility

Mobility is the measurement of the number of moves a player has. This feature is also used in other domains than Chess. In Chess, a player is checkmated when his King is threatened and he has no legal moves left. Additionally, the more moves are eligible, the higher the probability is to find a good move. Thus, mobility can be a useful feature in the domain of multi-player Chess. For regular Chess, Slater (1988) showed that there is a correlation between a player's mobility and the probability to win the game if the material balance of the players is even.
In the program, first the mobility of each player is computed. Afterwards, the rounded average of all players left in the game is calculated and subtracted from the mobility of each player. Finally, for each player, a function is used to convert this number into a value which is added to the score of the corresponding player. The function, which is derived from the mobility feature of REBEL (Schröder, 2007), is as follows:

$$f(mobility) = \begin{cases} \frac{mobility}{128}, & \text{if } |mobility| \leq 11 \\ \frac{2 \times mobility - 11}{128}, & \text{if } 12 \leq |mobility| \leq 15 \\ \frac{3 \times mobility - 26}{128}, & \text{if } 16 \leq |mobility| \leq 19 \\ \frac{4 \times mobility - 45}{128}, & \text{if } 20 \leq |mobility| \end{cases}$$

The values are precomputed and stored in an array to save computation time.

5.1.7 Pawn Formation

Pawns are generally regarded as stronger if they are connected. Schröder (2007) proposed to evaluate three different kinds of connections. (1) Pawns side by side (e.g., white Pawn ♙ on G4 and H4) get the

Attacking pieces	Value added to counter
	0
♙	0
♘	0
♘ & ♙	0
♖	0
♖ & ♙	0
♕	0
♔	0
♔ & ♙	0
♔ & ♘	0
♔ & ♘ & ♙	0
♖ & ♘	1
♖ & ♘ & ♙	1
♕ & ♙	1
♔ & ♖	1
♔ & ♖ & ♙	1
♕ & ♘	2
♕ & ♘ & ♙	2
♕ & ♖	2
♔ & ♖ & ♘	2
♔ & ♕	2
♕ & ♖ & ♘	3
♔ & ♕ & ♙	3
♔ & ♕ & ♘	3
♔ & ♕ & ♖	3

(a) The value of the attack pattern; Bishops and Knights are treated as the same piece

Counter	King's value
0	3.0
1	2.9921875
2	2.984375
3	2.9765625
4	2.953125
5	2.9296875
6	2.90234375
7	2.85546875
8	2.8046875
9	2.70703125
10	2.609375
11	2.51171875
12	2.4140625
13	2.31640625
14	2.21875
15	2.12109375
16	2.0234375
17	1.92578125
17	1.828125
18	1.73046875
19	1.6328125
20	1.53515625
21	1.4375
22	1.33984375
23	1.2421875
24	1.14453125
25	1.046875
26	0.94921875
27	0.8515625
28	0.75390625
≥ 29	0.65625

(b) Relative values of the King

Figure 5.4: King Safety

highest bonus. (2) If a Pawn has no such connection, it gets a bonus if it is covered by another Pawn. (3) If it is also not covered, it gets a bonus if it would be covered if another Pawn would move forward. The bonus for the Pawn depends on its position. Figure 5.7 illustrates the bonus for white Pawns ♙ for each position on the board. Subfigure 5.5(a) depicts the bonus for white Pawns ♙ side by side and Subfigure 5.5(b) depicts the bonus for the both other connections. The illustrated numbers are divided by 256.

5.1.8 Bishop Pair

Bishops are more valuable if a player has a Bishop pair instead of a single Bishop. Especially in the endgame, when only a few pieces are left on the board, Bishops are more valuable than Knights. In the program, the score of a player gets increased if the player has two Bishops left. The fewer pieces the opponents have on the board, the higher the bonus is. The bonus consists of two parts. The first one counts the piece value of all opponents' Queens, Rooks, Knights and Bishops. This value is converted into the first part of the player's bonus by a conversion table (see Subtable 5.6(a)). The second part of the

0	0	0	0	0	0	0	0	0	0	0	0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
0	0	0	64	64	64	64	64	64	64	64	0	0	0		0	0	0	48	56	64	64	64	64	56	48	0	0	0		
0	0	0	60	60	60	60	60	60	60	60	0	0	0		0	0	0	44	52	60	60	60	60	52	44	0	0	0		
0	64	56	56	56	56	56	56	56	56	56	56	64	0		0	64	40	40	48	56	56	56	56	48	40	40	64	0		
0	64	52	52	52	52	52	52	52	52	52	52	64	0		0	64	36	36	44	52	52	52	52	44	36	36	64	0		
0	64	48	48	48	48	48	48	48	48	48	48	64	0		0	64	32	32	40	48	48	48	48	40	32	32	64	0		
0	64	44	44	44	44	44	44	44	44	44	44	64	0		0	64	28	28	36	44	44	44	44	36	28	28	64	0		
0	64	40	40	40	40	40	40	40	40	40	40	64	0		0	64	24	24	32	40	40	40	40	32	24	24	64	0		
0	64	32	32	36	40	40	40	40	36	32	32	64	0		0	64	16	16	24	32	32	32	32	24	16	16	64	0		
0	64	20	20	24	32	32	32	32	24	20	20	64	0		0	64	12	12	16	20	20	20	20	16	12	12	64	0		
0	64	12	12	16	24	24	24	24	16	12	12	64	0		0	64	8	8	12	16	16	16	16	12	8	8	64	0		
0	0	0	4	4	4	4	4	4	4	4	0	0	0		0	0	0	4	4	4	4	4	4	4	4	0	0	0		
0	0	0	0	0	0	0	0	0	0	0	0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0		
0	0	0	0	0	0	0	0	0	0	0	0	0	0		0	0	0	0	0	0	0	0	0	0	0	0	0	0		

(a) Pawn-position values for Pawns side by side

(b) Pawn-position values for covered Pawns

Figure 5.5: Pawn Formation; Values have to be divided by 256

Opponents' pieces value	Bonus
0 - 11	0.5
12 - 20	0.375
21 - 26	0.25
27 - 36	0.125
37 - 46	0.0625
≥ 46	0.03125

(a) Bonus for opponents' piece value

Number opponents' Pawns	Bonus
0 - 3	0.5
4 - 5	0.375
6 - 7	0.25
8 - 9	0.125
10 - 11	0.0625
12 - 15	0.03125
16 - 19	0.015625
20 - 23	0.0078125
24	0

(b) Bonus for opponents' Pawns

Figure 5.6: Bonus values for the bishop pair feature

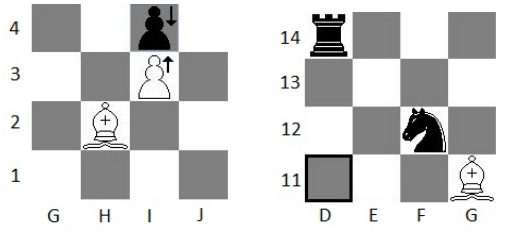
bonus takes the number of opponents' Pawns into account. Subtable 5.6(b) depicts the conversion from the number of opponents' Pawns to the bonus value. The two parts of the bonus are summed together and added to the score of the corresponding player.

5.1.9 Bad Bishop

A bad Bishop is a Bishop blocked by an own Pawn that cannot move easily. Figure 5.7(a) illustrates such a position. The white Pawn ♙ on I3 cannot move easily because the black Pawn ♟ on I4 prevents a forward move. Therefore the white Bishop ♗ is quite immobile in the next moves. The value by which the value of the Bishop ♗ is decreased depends on its position and on the blocking piece in front of the Pawn ♙. If this blocking piece is an own or an opponent's Pawn ♙ the value of the Bishop is decreased by 0.03125. If it is an opponent's Knight or Bishop the value gets decreased by 0.015625. For other blocking pieces, the value is decreased by 0.0078125. The position of the Bishop can multiply this value up to a factor of 4. All of these values are taken from the evaluation function of the regular Chess program REBEL (Schröder, 2007).

5.1.10 Pins

A pin is a position in which a defending piece cannot move without exposing a more valuable piece. Only Bishops, Rooks and Queens are able to pin opponent pieces. Figure 5.7(b) illustrates such a position. If the black Knight ♞ on F12 would move, the white Bishop ♗ on G11 could capture the black Rook ♜ on D14. The pseudo-code of this feature can be seen in Algorithm 5.1.2. There, the piece which pins (♗ in Figure 5.7(b)) is called *pinner*, the pinned piece (♞ in Figure 5.7(b)) is called *pinned* and the piece



(a) The white Bishop is blocked. (b) The black Knight is pinned.

Figure 5.7: Bad bishop and pins

behind the pinned piece (♜ in Figure 5.7(b)) is called *behind*. The calculated value is subtracted from the score of the player with the pinned piece.

Algorithm 5.1.2 Pseudo-code for the pin feature

```
1:  function pinFeature(pinner, pinned, behind, pinnedPlayer)
2:      if (behind is Rook, Queen or King)
3:
4:          if (pinner is Queen)
5:              score[pinnedPlayer] -= 0.0390625
6:              return
7:
8:          if (pinner is Rook)
9:
10:             if (pinned is Queen or King)
11:                 score[pinnedPlayer] -= 1.0
12:                 return
13:             else
14:                 score[pinnedPlayer] -= 0.15625
15:                 return
16:
17:
18:         if (pinner is Bishop)
19:
20:             if (pinned is Rook, Queen or King)
21:                 score[pinnedPlayer] -= 1.0
22:                 return
23:
24:             if (behind is Rook or King)
25:                 score[pinnedPlayer] -= 0.078125
26:                 return
27:             else
28:                 score[pinnedPlayer] -= 0.05859375
29:                 return
30:
31:
32:
33: end function
```

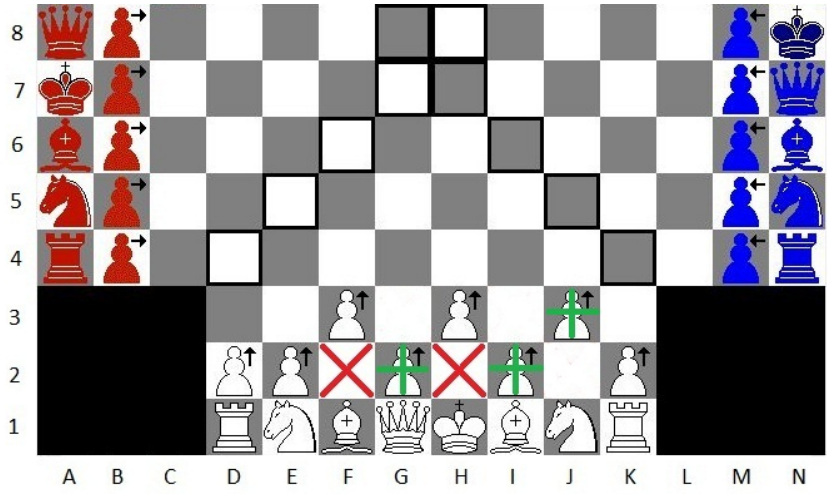


Figure 5.8: Pawns on the marked positions get a bonus/penalization

5.2 Opening

Two features of the evaluation function are only used in the beginning of a game. The first feature gives a bonus to Pawns on predefined positions and is described in Subsection 5.2.1. The second feature, described in Subsection 5.2.2, penalizes pieces in the centre of the board.

5.2.1 Position of the Pawns

In the first 80 moves of the game (20 moves per player if no one is defeated), the position of the Pawns are evaluated. In the following, this feature is described for White. Figure 5.8 depicts which squares increase respectively decrease the value of a white Pawn ♙ by 0.15. The Pawns ♙ on F2 and H2 should be moved in the beginning because this offers the Queen ♕ some possibilities to move. Furthermore, the Queen ♕ pins (see Subsection 5.1.10) the red Pawn ♟ on B6 and the blue Pawn ♟ on M7 and builds pressure on the corresponding Kings. The Pawns ♙ on G2 and I2 protect the King ♔ and therefore it is a bad decision to move them. The largest weakness of the defence is the square I2. The blue Queen ♛ can easily checkmate White when the square I2 is covered. Thus, it is important to move the Pawn ♙ to J3 to avoid the attack of the blue Queen ♛.

5.2.2 Dangerous Centre

In regular Chess it is important to control the centre of the board. Centre control increases the mobility (see Subsection 5.1.6) and offers possibilities to attack the opponent. In multi-player Chess it is risky to move a piece to the centre of the board because this piece can easily be attacked by all three opponent players. If e.g., White moves a Knight ♘ to the centre and the first opponent plays a move which threaten this Knight ♘, then the other opponents expect White to move the uncovered Knight ♘ and therefore they can attack White's defence. In the program, the centre is defined as the rectangular area consisting of the 16 squares from F6 to I9 and the value of a piece in this area gets decreased be 5% if there are 40 or more pieces on the board.

5.3 Lazy Evaluation

Computing all the features explained in this chapter costs some time. Only the two features *piece value* (see Subsection 5.1.1) and *en prise* (see Subsection 5.1.2) have a large influence on the score of each player. The value of the other features does not change that much during a move. Thus, the evaluation function in the program is divided into two stages. In the first stage, only the value of the both mentioned features are computed. In some positions, this value exceeds β by a given margin or it is lower than α minus the

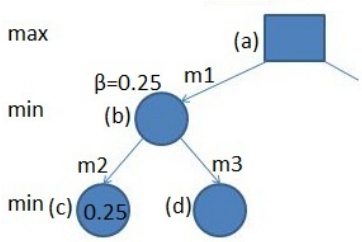


Figure 5.9: Lazy evaluation function

margin. In those cases, the second stage of the evaluation function which computes the value of the other features does not have to be started. Therefore, the lazy evaluation can increase the performance of the program.
For instance, if a paranoid player searches for the move combination of all opponents which decreases his value the most, then he does not have to compute the whole position after one opponent has captured another opponent's Queen. This capturing will decrease the value of one opponent a lot and therefore the score of the root player will increase. Figure 5.9 illustrates such a search. Consider all pieces are on the board (the en prise feature is not considered in this example). In this case, all players have the piece value of 43. After investigating Node *(c)*, the value of β at Node *(b)* is $\frac{43}{43\times 4} = 0.25$. Move *m3* captures an opponent's Queen. The score of the root player using only the piece value is

$$\frac{43}{4 \times 43 - 10} = 0.2654321.$$

All the other features will decrease the value of the root player at most by a margin, in the program the value of a Pawn. Therefore the expected value of the root player is at least

$$\frac{43 - 1}{4 \times 43 - 10 - 1} = 0.26087 > 0.25.$$

The value of this position exceeds β by the given margin and therefore the other features do not have to be computed. The lazy evaluation can also be used for α-prunings. In this case, the value of a Pawn is added to the value of the player instead of subtracted.

Chapter 6

Experiments and Results

In this chapter, the experiments performed with the program for playing multi-player Chess, developed in Java 1.6.0, and the corresponding results are presented. The settings of the experiments are explained in Section 6.1. The results of the experiments performed to test the quiescence search are shown in Section 6.2. In Section 6.3, the performance of the three existing algorithms, *paranoid*, max^n and *best-reply search*, are presented. The questions (see Section 4.2) regarding the move ordering in the proposed *best-reply* variants are answered by experiments in Section 6.4. Afterwards, the strength of the proposed *best-reply* variants are compared in Section 6.5. The strongest of the *best-reply* variants is compared to the existing algorithms *paranoid*, max^n and *best-reply* in Section 6.6. A conclusion of all the performed experiments is given in Section 6.7.
For all tables in this chapter, the numbers in the brackets indicate the win ratio, applied with a 95% confidence interval. The columns labelled with 'Time' state the time in milliseconds each player had for computing the move he considers best and the columns labelled with 'Games' indicate how many games were played.

6.1 Settings

Multi-player Chess is always played with four players. Table 6.1 shows all the possible settings for comparing two, three and four different algorithms against each other. The position of the players can influence their strength and therefore all possible combinations have to be used in the experiments to gain reliable results. For comparing two algorithms against each other, there exist 14 combinations (see Subtable 6.1(a)). Six of these 14 combinations consist of two players using one algorithm and two players using another algorithm. The eight remaining combinations consist of one player using one algorithm against three players using another algorithm. For comparing three algorithms against each other, one of the algorithms has to be played by two players. There exist 36 combinations as depicted in 6.1(b). The 24 combinations for comparing four algorithms against each other are depicted in 6.1(c).
In the max^n algorithm shallow pruning, described in Subsection 3.3.1, is applied. In all the other algorithms, $\alpha\beta$-pruning (see Subsection 3.1.1) is applied. In all algorithms, the search enhancements described in Section 3.2 are used for the searching players. To find out whether the *transposition tables*, *killer moves* and *history heuristics* are also useful for the non-searching opponents, experiments are performed in Subsection 6.4.3. All the experiments are performed on 64-Bit Linux machines with a speed of 2.4 GHz.

6.2 Quiescence Search

In this section, the usability of the quiescence search (Section 3.4) is tested for the three search algorithms *paranoid* (Subsection 6.2.1), max^n (Subsection 6.2.2) and *best-reply* (Subsection 6.2.3). Subsection 6.2.4 gives a summarization and an explanation of the results.

Pos. 1	Pos. 2	Pos. 3	Pos. 4
1	1	2	2
1	2	1	2
1	2	2	1
2	1	1	2
2	1	2	1
2	2	1	1
1	1	1	2
1	1	2	1
1	2	1	1
2	1	1	1
2	2	2	1
2	2	1	2
2	1	2	2
1	2	2	2

(a) Two-algorithm setting

Pos. 1	Pos. 2	Pos. 3	Pos. 4
1	1	2	3
1	1	3	2
1	2	1	3
1	3	1	2
1	2	3	1
1	3	2	1
2	1	1	3
3	1	1	2
2	1	3	1
3	1	2	1
2	3	1	1
3	2	1	1
2	2	1	3
2	2	3	1
2	1	2	3
2	3	2	1
2	1	3	2
2	3	1	2
1	2	2	3
3	2	2	1
1	2	3	2
3	2	1	2
1	3	2	2
3	1	2	2
3	3	2	1
3	3	1	2
3	2	3	1
3	1	3	2
3	2	1	3
3	1	2	3
2	3	3	1
1	3	3	2
2	3	1	3
1	3	2	3
2	1	3	3
1	2	3	3

(b) Three-algorithm setting

Pos. 1	Pos. 2	Pos. 3	Pos. 4
1	2	3	4
1	2	4	3
1	3	2	4
1	3	4	2
1	4	2	3
1	4	3	2
2	1	3	4
2	1	4	3
2	3	1	4
2	3	4	1
2	4	1	3
2	4	3	1
3	1	2	4
3	1	4	2
3	2	1	4
3	2	4	1
3	4	1	2
3	4	2	1
4	1	2	3
4	1	3	2
4	2	1	3
4	2	3	1
4	3	1	2
4	3	2	1

(c) Four-algorithm setting

Figure 6.1: Experiments settings

6.2.1 Paranoid

Table 6.1 presents the results of the experiments performed for *paranoid* against *paranoid* enhanced with the quiescence search. For each of the 14 combinations in a two-algorithm setting, 40 games were computed for 1 and for 5 seconds thinking time per move. As explained in Section 3.4, opponents' capturing moves against non-root-players are not considered in the *paranoid* quiescence search. The result of this experiment is that using quiescence search does not influence the strength of the paranoid player. Lorenz and Tscheuschner (2006) also performed experiments with the quiescence search in the domain of multi-player Chess. Although they used a thinking time of 180 seconds per move, their results are quite similar. *Paranoid* without quiescence search won 50.4% of their 93 performed games.

Table 6.1: Experiments paranoid quiescence search

Time	Games	Paranoid	Paranoid with QS
1000	560	284.500 (50.8% ± 4.2%)	275.500 (49.2% ± 4.2%)
5000	560	277.333 (49.5% ± 4.2%)	282.666 (50.5% ± 4.2%)

6.2.2 Max^n

The result of the experiment performed to find out whether the quiescence search is useful in max^n, is given in Table 6.2. With a thinking time of 1 second per move, 40 games per combination were performed, resulting in 560 games. The quiescence search reduced the playing strength significantly. The version without quiescence search wins more than 93% of all games. The max^n version of the quiescence search is probably not successful because for each player in the quiescence search, capturing moves against all other opponents are considered. This can lead to long capturing sequences which cost quite some computation time and therefore decrease the search depth. Lorenz and Tscheuschner (2006) also performed these experiments. Their result is that max^n without quiescence search won 69.1% (± 9.6%) of the games against max^n with quiescence search. They come to the same conclusion that quiescence search is not successful in max^n. Nevertheless, there is a large difference in the results. In the program used for this thesis, max^n with quiescence search won 93.3% of the games while in the experiments performed by Lorenz and Tscheuschner (2006) it won 69.1% of the games. There are three possible explanations for that difference. (1) Lorenz and Tscheuschner (2006) used a thinking time of 180 seconds per move, while in this program 1 second were used. (2) Lorenz and Tscheuschner (2006) did not state that they use different quiescence search versions. It can be assumed that they use the same quiescence search as in the *paranoid* algorithm while this program uses a max^n version of the quiescence search (see Section 3.4). (3) The programs use different evaluation functions. Especially, the *en prise* feature, which is not used by Lorenz and Tscheuschner (2006), can have a large influence on the performance of the quiescence search because it also considers possible capturings of pieces.

Table 6.2: Experiments max^n quiescence search

Time	Games	Max^n	Max^n with QS
1000	560	522.250 (93.3% ± 2.1%)	36.75 (6.7% ± 2.1%)

6.2.3 Best-Reply

Best-reply uses a quiescence search where all min-players are considered as one player. Table 6.3 shows that the *best-reply search* is more successful without quiescence search. It won 57.5% of the 1120 games (80 per combination) against BRS with quiescence search.

Table 6.3: Experiments best-reply quiescence search

Time	Games	BRS	BRS with QS
1000	1120	643.750 (57.5% ± 2.9%)	476.250 (42.5% ± 2.9%)

6.2.4 Results of the Quiescence Search Experiments

As explained in Section 3.4, the quiescence search has one disadvantage and one advantage. The disadvantage is the additionally required computation time to assign a value to the leaf nodes of the tree. This can lead to a lower search depth in the given amount of time. The advantage is the consideration of capturing sequences in the positions to evaluate and therefore to avoid bad moves. The possible capturings of pieces is also considered by the *en prise* feature (see Subsection 5.1.2) of the evaluation function. Therefore the advantage of the quiescence search seems not large enough to accept the drawback of the

additionally required computation time. The additional computation time needed in the max^n search and *best-reply search* is probably higher than for the *paranoid* search. The reason for that is that the *paranoid* quiescence search terminates when the opponent on turn has no capturing move against the root player while the opponents in the max^n quiescence search consider capturing moves against all other players. The *best-reply* quiescence search considers all opponents as one opponent and therefore it is sufficient that only one of the three opponents has capturing moves against the root player to continue the quiescence search. In conclusion, the *paranoid* quiescence search requires probably less additional computation time than the quiescence search of the other two search algorithms. The results of the experiments give evidence for this hypothesis because they have shown that the quiescence search decreases the strength of max^n and *best-reply* while the quiescence search does not influence the strength of *paranoid*. Therefore, the quiescence search is not considered in the remaining experiments.

6.3 Performance of the Existing Algorithms

In this section, the performance of the three existing algorithms *paranoid*, max^n and *best-reply search* are tested in the domain of multi-player Chess. First, all three algorithms play against each other, described in Subsection 6.3.1. Afterwards, two of these algorithms are compared to each other to find out whether one algorithm is able to exploit the weaknesses of another algorithm. The comparison of *paranoid* and *best-reply search* is described in Subsection 6.3.2, the comparison of *paranoid* and max^n is described in Subsection 6.3.3 and the comparison of *best-reply search* and max^n is described in Subsection 6.3.4. Finally, Subsection 6.3.5 summarizes the results of the performed experiments.

6.3.1 Paranoid vs. Max^n vs. Best-Reply

The first experiment to compare the strength of the three existing search algorithms in the domain of multi-player Chess is to let them play against each other in the three player setting (see Section 6.1). 40 games were played for each of the 36 possible combinations. Table 6.4 depicts that max^n performs not as well as the other two algorithms. As the confidence interval points out, the difference between *paranoid* and *best-reply* is not that large. *Paranoid* seems to perform best in this setting because it won over 50 games more than *best-reply*.

Table 6.4: Experiments paranoid vs. max^n vs. best-reply

Time	Games	Paranoid	Max^n	BRS
1000	1440	606.000 (42.1% ± 2.6%)	281.666 (19.6% ± 2.1%)	552.333 (38.4% ± 2.6%)

6.3.2 Paranoid vs. Best-Reply

To compare *paranoid* and *best-reply*, 40 games for each of the 14 combinations are played. The result is shown in Table 6.5. *Best-reply* performs better than *paranoid*. It won almost 59% of the games. This result is unexpected because *paranoid* was better in the three-algorithm setting explained in Subsection 6.3.1. It seems to be that *best-reply* is able to exploit the weaknesses of *paranoid* and *paranoid* is able to exploit the weaknesses of max^n.

Table 6.5: Experiments paranoid vs. best-reply

Time	Games	Paranoid	BRS
1000	560	230.166 (41.1% ± 4.1%)	329.833 (58.9% ± 4.1%)

6.3.3 Paranoid vs. Max^n

Table 6.6 presents the results of the games where *paranoid* played against *max^n*. *Paranoid* won 59% of the 1120 played games (80 games per combination). Lorenz and Tscheuschner (2006) also performed this experiment in the domain of multi-player Chess. Their result is that *paranoid* won 89.6% (± 10.4%) of the games against *max^n*. Thus, in their implementation *paranoid* performs much better against *max^n* than in this program. One reason for this difference could be the en prise feature (see Subsection 5.1.2). Lorenz and Tscheuschner (2006) only used piece value, mobility, king safety and piece-square tables as features of their evaluation function. The en prise feature makes *max^n* a bit more paranoid because it sometimes prevents the *max^n*-player to move a piece to a dangerous position. Experiments performed with an earlier version of the evaluation function which only consists of the piece value feature achieved similar results as Lorenz and Tscheuschner (2006). *Paranoid* won 83.0% (± 6.3%) of the games against *max^n*.

Table 6.6: Experiments paranoid vs. max^n

Time	Games	Paranoid	Max^n
1000	1120	660.833 (59.0% ± 2.9%)	459.166 (41.0% ± 2.9%)

6.3.4 Best-Reply vs. Max^n

Table 6.7 shows that the *best-reply search* performs better than *max^n* because *best-reply* won 56% of the 560 (40 per combination) played games. It performs similar as *paranoid* against *max^n*.

Table 6.7: Experiments best-reply vs. max^n

Time	Games	BRS	Max^n
1000	560	313.583 (56.0% ± 4.2%)	246.416 (44.0% ± 4.2%)

6.3.5 Results of the Comparison of the Existing Algorithms

The result of the performed experiments is that *max^n* performs not as well as the both other existing algorithms. It wins the fewest games in the three-algorithm setting and also win fewer games in a two-algorithm setting than the other algorithms. Neither of both remaining algorithms is the strongest one. In a three-algorithm setting where *paranoid*, *max^n* and *best-reply* play against each other, *paranoid* performed a bit better than *best-reply*. But in the direct comparison, *best-reply* is stronger than *paranoid*. A reason for the strong performance of *best-reply* against *paranoid* can be that *best-reply* is able to exploit the weaknesses of *paranoid*.

6.4 Move Ordering for the BMOMs in the Best-Reply Variants

In this section, the experiments regarding the move ordering of the non-searching opponents in the search of the proposed *best-reply* variants are described. As explained in Subsection 4.3.5, the move ordering is quite important for the success of the *best-reply* variants. First, experiments to find out whether it is more successful to use a *paranoid* or a *max^n* static move ordering for the non-searching opponents are described in Subsection 6.4.1. Subsequently, experiments with small differences in the used domain knowledge for the static move ordering are described in Subsection 6.4.2. In Subsection 6.4.3, experiments are performed to find out whether the dynamic move ordering techniques increase the performance of the *best-reply* variants. Finally, the results of all the experiments regarding the move ordering are summarized and the resulting configuration of the *best-reply* variants for the following experiments are explained in Subsection 6.4.4.

6.4.1 Max^n vs. Paranoid Move Ordering

As explained in Section 4.2, a *max^n* move ordering for the non-searching opponents can lead to bad moves. The alternative to use a *paranoid* move ordering can, similar to the *paranoid* algorithm, be a too pessimistic assumption. The difference between a *max^n* and a *paranoid* move ordering is that the *max^n* move ordering assigns a high value to capturing moves against all opponents while the *paranoid* move ordering assigns only a high value to capturing moves against the root player. The following experiments show which of the both move orderings is the better choice for the *best-reply* variants. The experiments are performed for $BRS_{0,C}$, $BRS_{1,C-1}$ and $BRS_{2,C-2}$. They cannot be performed for $BRS_{2,0}$ because in this variant no opponent plays a BMOM.

$BRS_{0,C}$

The results of the 560 performed games (40 per combination) for $BRS_{0,C}$ are presented in Table 6.8. The version with a static *max^n* move ordering for the non-searching opponents won 39.3% of the games. Thus, using a *paranoid* move ordering for the non-searching opponents is stronger for $BRS_{0,C}$. By using a *max^n* move ordering, the root player seems to be too optimistic, leading to bad moves as explained in Section 4.2.

Table 6.8: Results of the different static move orderings for the BMOM in $BRS_{0,C}$

Time	Games	$BRS_{0,C}$ (max^n MO)	$BRS_{0,C}$ (paranoid MO)
1000	560	220.333 (39.3% ± 4.1%)	359.666 (60.7% ± 4.1%)

$BRS_{1,C-1}$

Table 6.9 presents the result of 1120 performed games (80 per combination) where $BRS_{1,C-1}$ with a static *max^n* move ordering for the non-searching opponents played against $BRS_{1,C-1}$ with a static *paranoid* move ordering for the non-searching opponents. The *paranoid* move ordering is better because it won almost 58% of all games.

Table 6.9: Results of the different static move orderings for the BMOM in $BRS_{1,C-1}$

Time	Games	$BRS_{1,C-1}$ (max^n MO)	$BRS_{1,C-1}$ (paranoid MO)
1000	1120	474.000 (42.3% ± 2.9%)	646.000 (57.7% ± 2.9%)

$BRS_{2,C-2}$

Table 6.10 presents the results for $BRS_{2,C-2}$. The difference between the two move orderings is not significant. In comparison to the experiments performed with $BRS_{0,C}$ and $BRS_{1,C-1}$, it is interesting that the *paranoid* move ordering for the non-searching opponents does not outperform the *max^n* counterpart. One explanation for this could be that the root player in this algorithm is too pessimistic by assuming that the two searching opponents and also the non-searching opponent play against him. It is more pessimistic than in the other tested *best-reply* variants because more opponents perform a *paranoid* search. Letting the opponents perform a *paranoid* search can lead to more pessimistic moves than letting them play the most pessimistic moves from the move ordering.

Table 6.10: Results of the different static move orderings for the BMOM in $BRS_{2,C-2}$

Time	Games	$BRS_{2,C-2}$ (max^n MO)	$BRS_{2,C-2}$ (paranoid MO)
1000	560	284.500 (50.8% ± 4.2%)	275.500 (49.2% ± 4.2%)

Conclusion

Using a *paranoid* move ordering for the non-searching opponents leads to a better performance than using a max^n counterpart for $BRS_{0,C}$ and $BRS_{1,C-1}$. For $BRS_{2,C-2}$ the both ways to order the moves seems to be equally good. In the following experiments, the *paranoid* move ordering is used for all *best-reply* variants.

6.4.2 Different Static Move Orderings

The static move ordering used in the program only considers the captured piece (see Subsection 3.2.1). This is sufficient for the existing algorithms because in the most cases, the move ordering only influences the efficiency of the pruning and not the decision about the best move. In the *best-reply* variants where some opponents play a BMOM, the static move ordering defines the BMOM and therefore a different move ordering can change the decision of the root player. The move ordering is quite important for the performance of the *best-reply* variants. Thus, two other move orderings are tested to find out, whether one of them can increase the performance of the *best-reply* variants.

Consider Capturing Piece

Only considering the captured piece for the static move ordering has one disadvantage. If two pieces can attack the same opponent's piece, the first generated move gets ordered first. If, for instance, a white Queen ♕ and a white Pawn ♙ both can attack a black Knight ♞, then the move generated first is also ordered first, although in the most cases it is better to capture the Knight ♞ with the Pawn ♙ because the Knight ♞ can be covered and therefore a recapturing could happen. The static move ordering tested here considers also the value of the capturing piece. If two moves capture a same valued piece, the lower valued capturing piece gets ordered first. In the experiments presented in Table 6.11, the old static move ordering played against this new static move ordering. The used algorithm in the experiment was $BRS_{1,C-1}$. The new move ordering performs a bit better in the 1120 performed games, but as the confidence interval indicates, it cannot be said with certainty that the new move ordering is really stronger.

Table 6.11: Old static move ordering vs. static move ordering considering also the attacking piece

Time	Games	Captured piece considered	Captured and capturing piece considered
1000	1120	531.000 (47.4% ± 3.0%)	589.000 (52.6% ± 3.0%)

Difference Between Captured and Capturing Piece

The drawback of the static move orderings explained so far is that they are not able to detect whether a piece is covered or not. Thus, a move which captures a Queen gets always ordered before a move which captures a Rook. The idea of the move ordering presented here is to assume that all pieces are covered and that the opponent will perform a recapture. Thus, the value of the capturing piece is as important as the value of the captured piece. The proposal is to use the value of the captured piece minus the value of the capturing piece to order the moves. Table 6.12 presents the result for $BRS_{1,C-1}$ where the proposed static move ordering played against the static move ordering explained in Subsection 6.4.2. It was sufficient to only perform 280 games (20 per combination) to get the result that using the difference between the two piece involved in the capturing is not a good way to order the moves. This new static move ordering won less than 17% of all games. The weakness of this move ordering is that only capturing moves, where the value of the captured piece is higher than the value of the capturing piece, get a positive value. A Queen would never be expected to play a capturing move and a Pawn would never be expected to get captured.

Table 6.12: Experiment to compare the strength of the new static move orderings.

Time	Games	Captured and capturing piece considered	Difference between both pieces
1000	280	234.416 (83.7% ± 4.4%)	45.583 (16.3% ± 4.4%)

Results of Different Static Move Orderings

The first proposed new static move ordering which also considers the value of the capturing piece seems to be the strongest static move ordering. It performs a bit better than the static move ordering which only considers the value of the captured piece. This new static move ordering is used to order the moves of the non-searching opponents in the following experiments.

6.4.3 Dynamic Move Ordering Techniques

This subsection presents the results of the performed experiments to find out whether the dynamic move ordering techniques increase the performance of the *best-reply* variants when they are used to order the moves of the non-searching opponents. For all the experiments in this subsection, the algorithm $BRS_{1,C-1}$ is used and 560 games are played (40 per combination).
Table 6.13 presents that there is no difference in the performance of the algorithm $BRS_{1,C-1}$ when transposition tables are used. As Table 6.14 presents, there is also no difference when killer moves are used. For the history heuristic, there is also no significant difference in the performance, as Table 6.15 presents. All three dynamic move ordering techniques does not significantly influence the strength of the *best-reply* variant $BRS_{1,C-1}$ when they are used to order the moves of the non-searching opponents. Therefore in all following tests, the dynamic move ordering techniques are not used for the non-searching opponents.

Table 6.13: Static move ordering against static move ordering plus TT

Time	Games	$BRS_{1,C-1}$ (static MO)	$BRS_{1,C-1}$ (static MO + TT)
1000	560	278.000 (49.6% ± 4.2%)	282.000 (50.4% ± 4.2%)

Table 6.14: Static move ordering against static move ordering plus KM

Time	Games	$BRS_{1,C-1}$ (static MO)	$BRS_{1,C-1}$ (static MO + KM)
1000	560	284.500 (50.8% ± 4.2%)	275.500 (49.2% ± 4.2%)

Table 6.15: Static move ordering against static move ordering plus HH

Time	Games	$BRS_{1,C-1}$ (static MO)	$BRS_{1,C-1}$ (static MO + HH)
1000	560	289.500 (51.7% ± 4.2%)	270.500 (48.3% ± 4.2%)

6.4.4 Results of the Move Ordering Experiments

There are three conclusions of the performed experiments about the move ordering of the non-searching opponents. (1) The *best-reply* variants perform better when they use a *paranoid* instead of a max^n move ordering for the non-searching opponents. (2) It is also a bit better to consider the capturing piece in the static move ordering. (3) The dynamic move ordering techniques does not significantly influence the strength of the player when they are applied to the move ordering of the non-searching opponents.

6.5 Performance of the Best-Reply Variants

In this section, the proposed *best-reply* variants are tested against each other to find out which of them is the strongest one. First, the experiment where all four variants play against each other in a four-algorithm setting is described in Subsection 6.5.1. Experiments where the algorithms play against each other in a three-algorithm setting are presented in Subsection 6.5.2. Subsequently, all the proposed algorithms are directly compared to each other in a two-algorithm setting in Subsection 6.5.3. The conclusion of these experiments is given in Subsection 6.5.4.

6.5.1 All Variants Against Each Other

Table 6.16 presents the result of 960 performed games (40 per combination) where all the proposed *best-reply* variants play against each other in a four-algorithm setting. $BRS_{1,C-1}$ is the strongest of the proposed algorithms because it won more than 41% of all games. The second best proposed algorithm is $BRS_{2,C-2}$ with a win ratio of 32.5%. $BRS_{2,0}$ won almost 26% of the played games. The unambiguous weakest of the proposed algorithms is $BRS_{0,C}$. It won less than 1% of all games. One explanation for this poor performance can be that this player does not expect his opponents to play tactical moves. Therefore, this player is often not able to detect opponents' moves which will checkmate him.

Table 6.16: $BRS_{1,C-1}$ vs. $BRS_{0,C}$ vs. $BRS_{2,0}$ vs. $BRS_{2,C-2}$

Time	Games	$BRS_{1,C-1}$	$BRS_{0,C}$	$BRS_{2,0}$	$BRS_{2,C-2}$
1000	960	396 (41.3% ± 3.2%)	6 (0.6% ± 0.5%)	246 (25.6% ± 2.8%)	312 (32.5% ± 3.0%)

6.5.2 Comparison of BRS-Variants in a Three-Algorithm Setting

Table 6.17 presents the results of the experiments performed in a three-algorithm setting. $BRS_{1,C-1}$ is the strongest of the proposed algorithms. In each of the three experiments with $BRS_{1,C-1}$, it has a win ratio of more than 49%. $BRS_{0,C}$ is clearly the weakest of the proposed algorithms. $BRS_{2,0}$ performs a bit better than $BRS_{2,C-2}$.

Table 6.17: Comparison of the proposed best-reply variants in a three-algorithm setting

Time	Games	$BRS_{1,C-1}$	$BRS_{0,C}$	$BRS_{2,0}$	$BRS_{2,C-2}$
1000	576	307.000 (53.3% ± 4.1%)	15.500 (2.7% ± 1.4%)	253.500 (44.0% ± 4.1%)	-
1000	576	316.500 (54.9% ± 4.1%)	10.000 (1.8% ± 1.1%)	-	249.500 (43.3% ± 4.1%)
1000	576	286.000 (49.7% ± 4.1%)	-	159.500 (27.7% ± 3.7%)	130.500 (22.7% ± 3.5%)
1000	576	-	7.333 (1.3% ± 1.0%)	298.666 (51.9% ± 4.1%)	270.000 (46.9% ± 4.1%)

6.5.3 Comparison of BRS-Variants in a Two-Algorithm Setting

Experiments in a two-algorithm setting are performed to compare two of the proposed algorithms directly. The results of all possible experiments to compare the proposed algorithms are presented in Table 6.18. 560 games with thinking time of 1 second were played for each comparison. $BRS_{1,C-1}$ outperforms all the other algorithms. It wins 98% of the games against $BRS_{0,C}$, 67% of the games against $BRS_{2,0}$ and 62% of the games against $BRS_{2,C-2}$. $BRS_{0,C}$ is the weakest of the proposed algorithms with a win ratio of less than 10% against each of the other algorithms. The comparison of $BRS_{2,C-2}$ and $BRS_{2,0}$ is more difficult. In the direct comparison, they are almost equally strong. Against $BRS_{1,C-1}$, $BRS_{2,C-2}$ performs a bit better than $BRS_{2,0}$, but against $BRS_{0,C}$, $BRS_{2,0}$ performs better than $BRS_{2,C-2}$. Therefore, $BRS_{2,C-2}$ and $BRS_{2,0}$ are almost equally strong in a two-algorithm setting in the domain of multi-player Chess.

Table 6.18: Comparison of the proposed best-reply variants in a two-algorithm setting

	$BRS_{1,C-1}$	$BRS_{0,C}$	$BRS_{2,0}$	$BRS_{2,C-2}$
$BRS_{1,C-1}$	-	549.333 (98.1% ± 1.2%)	373.500 (66.7% ± 4.0%)	345.333 (61.7% ± 4.1%)
$BRS_{0,C}$	10.666 (1.9% ± 1.2%)	-	33.833 (6.0% ± 2.0%)	50.166 (9.0% ± 2.4%)
$BRS_{2,0}$	186.500 (33.3% ± 4.0%)	526.166 (94.0% ± 2.0%)	-	278.500 (49.8% ± 4.2%)
$BRS_{2,C-2}$	214.666 (38.3% ± 4.1%)	509.833 (91.0% ± 2.4%)	281.500 (50.2% ± 4.2%)	-

6.5.4 Conclusion

$BRS_{1,C-1}$ is the strongest of the proposed algorithms in the domain of multi-player Chess. It performed best in a four-algorithm setting against all the other proposed algorithms. In the three-algorithm setting it was also the strongest algorithm. Additionally, $BRS_{1,C-1}$ outperforms the other proposed *best-reply* variants in the direct comparison. $BRS_{2,C-2}$ and $BRS_{2,0}$ are almost equally strong. $BRS_{2,C-2}$ performed a bit better in the four-algorithm setting, but in a three-algorithm setting $BRS_{2,0}$ performed a bit better. In a two-algorithm setting they seem to be equally strong. The weakest of the proposed *best-reply* variants is clearly $BRS_{0,C}$. Neither in the four-algorithm setting nor in the three- and two-algorithm settings it won more than 9% of the games.

6.6 $BRS_{1,C-1}$ Against Existing Algorithms

$BRS_{1,C-1}$ is the strongest of the proposed *best-reply* variants. But only the comparison to the existing algorithms examines whether $BRS_{1,C-1}$ is a promising algorithm. It is first tested against all the existing algorithms in a four-algorithm setting in Subsection 6.6.1. In Subsection 6.6.2 the proposed algorithm is tested in a three-algorithm setting against *paranoid* and max^n. Afterwards it is tested against all the existing algorithms in a two-algorithm setting. First, the results of the experiments against *paranoid* are presented in Subsection 6.6.3. The experiment against max^n is described in Subsection 6.6.4 and finally, the experiment against the *best-reply search* is described in Subsection 6.6.5.

6.6.1 $BRS_{1,C-1}$ Against All the Existing Algorithms

In the first experiment performed to get an impression about the strength of $BRS_{1,C-1}$, the existing algorithms *best-reply*, max^n, *paranoid* and the proposed algorithm $BRS_{1,C-1}$ play against each other in a four-algorithm setting. For 1 and 5 seconds, 40 games were played per combination, resulting in 960 games. Table 6.19 presents the result of this experiment. $BRS_{1,C-1}$ performs better than the existing algorithms. With a computation time of 5 seconds, it won 35.4% of all games. As already mentioned in Section 6.3, *best-reply* and *paranoid* are comparably strong. In the setting with 1 second thinking time, *best-reply* performed a bit better, but *paranoid* performed better in the setting with 5 seconds thinking time. *Best-reply* seems to perform worse when the thinking time increases. With a thinking time of 1 second, it won 26.5% of the games while it only won 22.7% of the games with a thinking time of 5 seconds. The weakest of the four algorithms is max^n.

Table 6.19: $BRS_{1,C-1}$ vs. BRS vs. Max^n vs. Paranoid

Time	Games	$BRS_{1,C-1}$	BRS	Max^n	Paranoid
1000	960	316 (32.9% ± 3.0%)	254.5 (26.5% ± 2.8%)	170 (17.7% ± 2.5%)	219.5 (22.9% ± 2.7%)
5000	960	340 (35.4% ± 3.1%)	218 (22.7% ± 2.7%)	167 (17.4% ± 2.4%)	235 (24.5% ± 2.8%)

6.6.2 $BRS_{1,C-1}$ vs. Paranoid vs. Max^n

Table 6.20 presents the result of the experiment where $BRS_{1,C-1}$ plays against *paranoid* and max^n in a three-algorithm setting. $BRS_{1,C-1}$ outperforms the other two algorithms. It won almost every second game. *Paranoid* has a win ratio of 33.1% and max^n a win ratio of 17.6%. This experiment can be compared to the experiment performed in Subsection 6.3.1 where *best-reply* played against *paranoid* and

max^n. In that experiment, *best-reply* won 38.4% of the games. In conclusion, $BRS_{1,C-1}$ won exactly 11 percentage points more games against *paranoid* and *max*^{n} than *best-reply*. Thus, letting the non-searching opponents play a BMOM instead of passing leads to an considerable gain in performance. It is remarkable that *paranoid* wins 8.9 percentage points (from 42% to 33.1%) fewer games than in the experiment described in Subsection 6.3.1, while *max*^{n} only wins 2 percentage points (from 19.6% to 17.6%) fewer games. Compared to *best-reply*, $BRS_{1,C-1}$ seems especially to perform well against *paranoid*.

Table 6.20: $BRS_{1,C-1}$ against paranoid against max^n

Time	Games	$BRS_{1,C-1}$	Paranoid	Max^n
1000	720	355.333 (49.4% ± 3.7%)	238.333 (33.1% ± 3.5%)	126.333 (17.6% ± 2.8%)

6.6.3 $BRS_{1,C-1}$ vs. Paranoid

Table 6.21 presents the result of the 560 performed games where $BRS_{1,C-1}$ played against *paranoid*. $BRS_{1,C-1}$ outperforms *paranoid* with a win ratio of 70%. *Best-reply* won 59% of the games against *paranoid* (see Subsection 6.3.2). This confirms the hypothesis from Subsection 6.6.2 that $BRS_{1,C-1}$ performs especially well against *paranoid*.

Table 6.21: $BRS_{1,C-1}$ against paranoid

Time	Games	$BRS_{1,C-1}$	Paranoid
1000	560	391.666 (70.0% ± 3.8%)	168.333 (30.0% ± 3.8%)

6.6.4 $BRS_{1,C-1}$ vs. Max^n

The result of the 560 performed games $BRS_{1,C-1}$ against *max*^{n} are presented in Table 6.22. $BRS_{1,C-1}$ wins 64% of all games and is according to that stronger than *max*^{n}. $BRS_{1,C-1}$ performs also better against *max*^{n} than *best-reply* did. *Best-reply* won 56% of the games against *max*^{n} (see Subsection 6.3.4).

Table 6.22: $BRS_{1,C-1}$ against max^n

Time	Games	$BRS_{1,C-1}$	Max^n
1000	560	360.333 (64.3% ± 4.0%)	199.666 (35.7% ± 4.0%)

6.6.5 $BRS_{1,C-1}$ vs. BRS

Table 6.23 presents that $BRS_{1,C-1}$ performs better than *best-reply*. It won 54.2% of the 1120 performed games with 1 second thinking time per move. With a thinking time of 5 seconds, it won 57.5% of the games. As stated in Subsection 6.6.1, *best-reply* seems to perform worse when the thinking time increases. The result of these experiments is that letting the non-searching opponents play the BMOM instead of passing seems to be a good improvement of the *best-reply search*.

Table 6.23: $BRS_{1,C-1}$ against BRS

Time	Games	$BRS_{1,C-1}$	BRS
1000	1120	607.333 (54.2% ± 3.0%)	512.666 (45.8% ± 3.0%)
5000	560	322.000 (57.5% ± 4.1%)	238.000 (42.5% ± 4.1%)

6.6.6 Performance of $BRS_{1,C-1}$ with a Max^n Move Ordering for the BMOMs

The experiment presented in Table 6.9 indicates that $BRS_{1,C-1}$ with a *paranoid* move ordering for the non-searching opponents is stronger than $BRS_{1,C-1}$ with a max^n move ordering for the non-searching opponents. In some non-capturing games, it is difficult or not possible to find opponents' moves bad for the root player without performing a search. For such games, it has to be ckecked whether $BRS_{1,C-1}$ is also able to outperform the existing algorithms, especially *best-reply*. To give an indication how the proposed algorithm performs in such games, experiments with a max^n move ordering for the non-searching opponents are performed. The results of these experiments are presented in Table 6.24. For each comparison, 560 games with a computation time of 1 second per move are performed. The results of the experiments where $BRS_{1,C-1}$ with a *paranoid* move orering for the non-searching opponents played against the existing algorithms are also presented in this table. The performance of $BRS_{1,C-1}$ decreases when using a max^n move ordering for the non-searching opponents. Against *paranoid*, the win ratio decreases from 70% to 56.3% and against max^n, it decreases from 64.3% to 59.3%. The win ratio also decreases from 54.2% to 53.0% against *best-reply*. Nevertheless, $BRS_{1,C-1}$ with a max^n move ordering for the non-searching opponents is still stronger than the existing algorithm. Thus, $BRS_{1,C-1}$ can also be a suitable algorithm for domains where it is not possible to find opponents' moves bad for the root player without performing a search.

Table 6.24: Comparison of different $BRS_{1,C-1}$ move ordering variants against existing Algorithms

	Paranoid	Max^n	BRS
$BRS_{1,C-1}$ max^n BMOMs	56.3% (± 4.2%)	59.3% (± 4.1%)	53.0% (± 4.2%)
$BRS_{1,C-1}$ paranoid BMOMs	70.0% (± 3.8%)	64.3% (± 4.0%)	54.2% (± 3.0%)

6.6.7 Conclusion of the Comparison to the Existing Algorithms

$BRS_{1,C-1}$ is a promising algorithm. In a four-algorithm setting against *best-reply*, *paranoid* and max^n, it performs better than the other algorithms. It is also better than each of the existing algorithms in the direct comparison. It performs especially well against *paranoid* with a win ratio of 70%.

6.7 Conclusion of the Experiments

The experiments in Section 6.2 pointed out that quiescence search is not useful in the domain of multi-player Chess. Regarding the move ordering of the non-searching opponents in the proposed *best-reply* variants, the experiments in Section 6.4 revealed that it is good to use a *paranoid* static move ordering. Applying dynamic move ordering techniques does not influence the strength of the variants. The conclusion of Section 6.5 is that $BRS_{1,C-1}$ is the best of the proposed algorithms and Section 6.6 pointed out that this algorithm performs better than the existing algorithms. Therefore, the most important conclusion of the experiments is that the proposed algorithm $BRS_{1,C-1}$ is a promising algorithm which outperforms the existing algorithms in the domain of multi-player Chess.

Chapter 7

Conclusions and Future Research

In this chapter, the conclusion of this thesis is given. First, the research questions are answered in Section 7.1. Afterwards the problem statement is answered in Section 7.2. Finally, the future research is given in Section 7.3.

7.1 Answering the Research Questions

In this section, the two research question stated in Section 1.3 are recalled and answered.

1. Is the best-reply search better than the well-known paranoid and max^n algorithms in the domain of non-cooperative multi-player Chess?

This questions is answered by the experiments described in Section 6.3. *Best-reply* performs better than *max*^{n}. It won 56% of the games against *max*^{n} in the direct comparison. It also performs better than *max*^{n} in the three-player setting with *paranoid* as third algorithm. The comparison of the *best-reply search* and the *paranoid* algorithm is more complicated. In the three-player setting mentioned before, *paranoid* performs a bit better than *best-reply* (42% to 38%). In contrast to this result, *best-reply* performs well against *paranoid* in the direct comparison. It wins almost 59% of these games. It is remarkable that *best-reply* performs better against *paranoid* (win ratio 59%) than against *max*^{n} (win ratio 56%) in the direct comparison. In conclusion, *best-reply* outperforms *max*^{n} and has a similar strength as *paranoid* in the domain of multi-player Chess.

2. Can we find variants of the best-reply search that are even better?

The conclusion of the experiments performed in Section 6.5 is that $BRS_{1,C-1}$ is definitely the strongest of the proposed *best-reply* variants. Further, $BRS_{1,C-1}$ also outperforms the existing algorithms in the domain of multi-player Chess. Against *best-reply*, it has a win ratio of 54.2% with a thinking time of 1 second and a win ratio of 57.5% with a thinking time of 5 seconds. It has a win ratio of 70% against *paranoid* and 64% against *max*^{n}. Thus, $BRS_{1,C-1}$ is a promising algorithm which performs better than the existing algorithms in the domain of multi-player Chess.

7.2 Answering the Problem Statement

The problem statement of the thesis is answered in this section.

How can the best-reply search be modified to outperform the paranoid algorithm in complex (capturing) multi-player games?

The most promising modification of *best-reply search* is to let the non-searching opponents play the strongest move based on the static move ordering instead of passing. The static move ordering is quite important for this modification because the best move regarding this move ordering is the root player's expectation of the moves played by the non-searching opponents. A bad static move ordering causes the algorithm to play bad moves. This modification solves the main problem of *best-reply* that passing

can lead to officially unreachable and in some domains illegal board positions. The result of the best-case analysis performed in Section 4.4 is that this modification decreases the number of leaf nodes a bit while the number of internal nodes increase. In the domain of multi-player Chess, the number of leaf nodes is more important than the number of internal nodes because the evaluation function is more time consuming than the move generation.
The experiment performed with the modified algorithm, called $BRS_{1,C-1}$, indicates that this algorithm is promising. It outperforms the existing algorithms *paranoid*, max^n and *best-reply*. It performs especially well against *paranoid*.

7.3 Future Research

In this section suggestions for the future research are given. These suggestions can be categorized into seven categories.

1. **Evaluation function**
   Additional work can be performed on the evaluation function to increase the playing strength. Although the evaluation function used in the program has more than 10 features, even more features can be implemented. For regular Chess there are many features that can be adapted to multi-player Chess to increase the playing strength. Especially, features representing endgame knowledge can be beneficial because no such feature is included in the current evaluation function. Additionally, experts for multi-player Chess can be searched to retrieve domain knowledge that can be used to add some more features to the evaluation function. Currently, the evaluation function uses two features for the opening. The performance in this part of the game can be increased by building and using opening books. The weights of the different features are tuned by hand or by just a few experiments. Therefore the playing strength can be increased by using machine learning techniques to tune the existing parameters of the evaluation function.

2. **Forward pruning**
   Forward pruning techniques are not used in the program. In quite some regular Chess programs, forward pruning techniques are applied. Therefore, it can be investigated whether applying forward pruning techniques like *Null Move* (Donninger, 1993), *ProbCut* (Buro, 1995) and *MultiCut* (Björnsson and Marsland, 1999) can increase the playing strength.

3. **$\alpha\beta$-enhancements**
   The $\alpha\beta$-framework can be enhanced by several techniques like *MTD(f)* (Plaat *et al.*, 1995) or *Principal Variation Search* (Fishburn, Finkel, and Lawless, 1980; Marsland and Campbell, 1982)/*NegaScout* (Reinefeld, 1983). This can also increase the playing strength.

4. **Faster framework**
   As part of this thesis, a framework was implemented from scratch. This was the first time that the author of this thesis implemented a framework for playing a multi-player game. A few weeks were spent to optimize the framework, but nevertheless the framework can be even more efficient by changing some implementation details like using *bitboards* (Heinz, 1997; Hyatt, 1999; Reul, 2009) to represent the board.

5. **Static move ordering**
   As explained in Subsection 4.3.5, the static move ordering is quite important for the performance of the proposed algorithms. The static more ordering in the program is quite simple. Spending more time to enhance the static move ordering could even increase the performance of $BRS_{1,C-1}$. One improvement can be to consider check moves. This is not considered in the program because it cost some computation time to identify check moves. But for the static move ordering for the non-searching opponents, this can be a promising approach to improve the quality of the move ordering. Another approach to improve the static move ordering can be the use of piece-square tables. Piece-square tables can be used to evaluate the position of the piece to move with respect to the kind of the moved piece.

6. **Allow non-searching opponents to investigate more than one move**
   In the proposed algorithm $BRS_{1,C-1}$ all but one opponents play the best move regarding the static

move ordering. When check moves are also considered in the static move ordering as proposed above, the question arises whether it is better to expect the opponent to play a good capture move or a check move. It could be a promising idea to propose an algorithm where the "non-searching opponents" are allowed to perform a search with a small branching factor like 2 or 3. In such an algorithm it would be possible to consider check moves and also capture moves for the non-searching opponents. This modification would increase the complexity of the tree, but it would still be much smaller than a *paranoid*-tree. It has to be tested whether the better estimation of the opponents' moves is worth the higher complexity of the tree.

7. **Other domains**
   The proposed algorithms are only tested in the domain of multi-player Chess. It has to be tested whether the best of the proposed algorithms, $BRS_{1,C-1}$, is also able to outperform the existing algorithms in other domains. The algorithm can be tested in other capture games and also in non-capture games. Additionally, it can be tested in games with imperfect information. It is from special interest to see how the properties of the different games have an influence on the performance of $BRS_{1,C-1}$. In multi-player Chess, the evaluation function is more time consuming than the move generation. In other games where the move generation is more complex or the board position is quite easy to evaluate, the larger number of internal nodes compared to *best-reply* can be such a disadvantage that it is not worth to let the non-searching opponents play a move. Moreover, in multi-player Chess the opponents' moves have a quite large influence on the position of the root player. Thus, it is quite important to let each opponent play a move. For other games where the opponents' moves does not have such a large influence on the root player's position, for instance racing-games, it has to be tested whether $BRS_{1,C-1}$ is also able to outperform *best-reply*.

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Appendix A

Algorithms

A.1 *Minimax* and *paranoid* with $\alpha\beta$-pruning

The pseudo-code for *minimax* and *paranoid* is the same. Therefore it is listed only once.

Algorithm A.1.1 Pseudo-code for the minimax-algorithm with $\alpha\beta$-pruning

```
1:  function alphaBeta(node, depth, alpha, beta)
2:      if (depth == 0 or node is terminal)
3:          return evaluate(node)
4:
5:      if (node is max-node)
6:          maxValue = −∞
7:          ForEach (child of node)
8:              maxValue = max(maxValue, alphaBeta(child, depth-1, alpha, beta))
9:              if (maxValue > alpha)
10:                 alpha = maxValue
11:                 if (alpha ≥ beta)
12:                     return maxValue
13:
14:
15:         return maxValue
16:
17:     if (node is min-node)
18:         minValue = ∞
19:         ForEach (child of node)
20:             minValue = min(minValue, alphaBeta(child, depth-1, alpha, beta))
21:             if (minValue < beta)
22:                 beta = minValue
23:                 if (alpha ≥ beta)
24:                     return maxValue
25:
26:
27:         return minValue
28:
29: end function
```

A.2 *Max*^{n} with shallow pruning

The variable maxSum in the pseudo-code is the maximum sum available in this game for all players. In multi-player Chess the winner gets 1 point, or in case of a draw the 1 point is shared by the remaining players. Therefore the maxSum in multi-player Chess is 1.

Algorithm A.2.1 Pseudo-code for the max^n-algorithm

```
1: function MAXN(node, depth, pruneValue)
2:     if (depth == 0 or node is terminal)
3:         return evaluate(node)
4:     maxValues = maxN(firstChild, depth-1, −∞)
5:     ForEach (child of node)
6:         values = maxN(child, depth-1, maxValues[playerOnTurn])
7:         if (values[playerOnTurn] > maxValues[playerOnTurn])
8:             maxValues = values
9:             if (maxValues[playerOnTurn] ≥ maxSum − pruneValue)
10:                break
11:
12:
13:    return maxValues
14: end function
```

A.3 *Best-reply search* with $\alpha\beta$-prunging

Algorithm A.3.1 Pseudo-code for the best-reply search

```
1:  function bestReply(node, depth, maxPlayer, alpha, beta)
2:      if (depth == 0 or node is terminal)
3:          return evaluate(node)
4:
5:      if (maxPlayer)
6:          maxValue = −∞
7:          ForEach (child of node)
8:              maxValue = max(maxValue, bestReply(child, depth-1, false, alpha, beta))
9:              if (maxValue > alpha)
10:                 alpha = maxValue
11:                 if (alpha ≥ beta)
12:                     return maxValue
13:
14:
15:         return maxValue
16:
17:     if (not maxPlayer)
18:         minValue = ∞
19:         childs = getChildsOfAllMinPlayers(node)
20:         ForEach (childs)
21:             minValue = min(minValue, bestReply(child, depth-1, true, alpha, beta))
22:             if (minValue < beta)
23:                 beta = minValue
24:                 if (alpha ≥ beta)
25:                     return maxValue
26:
27:
28:         return minValue
29:
30: end function
```