Tutorial on Epistemic Game Theory
Part 1: Static Games

Andrés Perea

EPICENTER
Research Center for
Epistemic Game Theory

Maastricht University

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Introduction

- Game theory studies situations where you make a decision, but where the final outcome also depends on the choices of others.
- Before you make a choice, it is natural to reason about your opponents – about their choices but also about their beliefs.
- Oskar Morgenstern, in 1935, already stresses the importance of such reasoning for games.

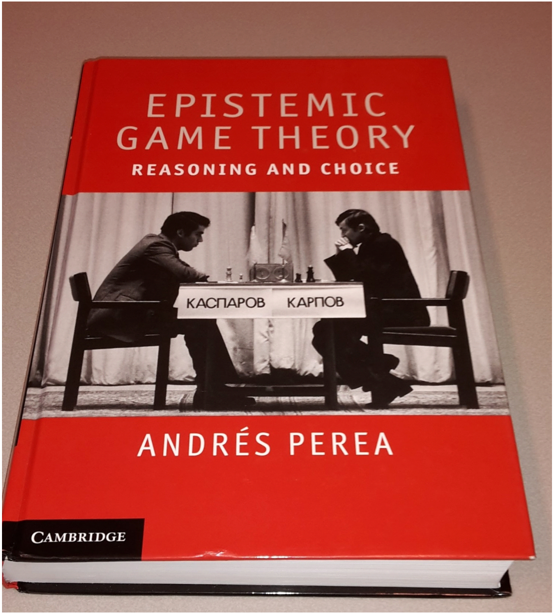
- Classical game theory has focused mainly on the choices of the players.
- Epistemic game theory asks: Where do these choices come from?
- More precisely, it studies the beliefs that motivate these choices.
- Since the late 80's it has developed a broad spectrum of epistemic concepts for games.
- Some of these characterize existing concepts in classical game theory, others provide new ways of reasoning.

Outline

- In the first part, we focus on static games.
- We discuss, and formalize, the idea of common belief in rationality.
- We present a recursive procedure to compute the induced choices .
- We provide an epistemic foundation for Nash equilibrium, and see that it requires more than just common belief in rationality.
- We investigate the extra conditions that lead to Nash equilibrium.

Outline

- In the second part, we move to dynamic games.
- We will see that the idea of common belief in rationality can be extended in at least two different ways to dynamic games:
- backward induction reasoning, leading to common belief in future rationality.
- forward induction reasoning, leading to common strong belief in rationality.
- We present both concepts formally.
- We provide recursive procedures for both concepts.


EPISTEMIC
GAME THEORY
REASONING AND CHOICE
КАСПАРОВ
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Common belief in rationality

Idea

- If you are an expected utility maximizer, you form a belief about the opponents' choices, and make a choice that is optimal for this belief.
- That is, you choose rationally given your belief.
- It seems reasonable to believe that your opponents will choose rationally as well, ...
- and that your opponents believe that the others will choose rationally as well, and so on.
- Common belief in rationality.

Example: Going to a party

	blue	green	red	yellow	same color as friend
you	4	3	2	1	0
Barbara	2	1	4	3	0

Story

- This evening, you are going to a party together with your friend Barbara.
- You must both decide which color to wear: blue, green, red or yellow.
- Your preferences for wearing these colors are as in the table. These numbers are called utilities.
- You dislike wearing the same color as Barbara: If you both would wear the same color, your utility would be 0.
- What color would you choose, and why?

	blue	green	red	yellow	same color as friend
you	4	3	2	1	0
Barbara	2	1	4	3	0

- Choosing blue is optimal if you believe that Barbara chooses green.
- Choosing green is optimal if you believe that Barbara chooses blue.
- Choosing red is optimal if you believe that, with probability 0.6, Barbara chooses blue, and that with probability 0.4 she chooses green.
- Hence, blue, green and red are rational choices for you.

	blue	green	red	yellow	same color as friend
you	4	3	2	1	0
Barbara	2	1	4	3	0

- Choosing yellow can never be optimal for you, even if you hold a probabilistic belief about Barbara's choice.
- If you assign probability less than 0.5 to Barbara's choice blue, then by choosing blue yourself, your expected utility will be at least $(0.5) \cdot 4 = 2$.
- If you assign probability at least 0.5 to Barbara's choice blue, then by choosing green yourself your expected utility will be at least $(0.5) \cdot 3 = 1.5$.
- Hence, whatever your belief about Barbara, you can always guarantee an expected utility of at least 1.5.
- So, yellow can never be optimal for you, and is therefore an irrational choice for you.

	blue	green	red	yellow	same color as friend
you	4	3	2	×	0
Barbara	2	1	4	3	0

- If you believe that Barbara chooses rationally, and believe that Barbara believes that you choose rationally,

  then you believe that Barbara will not choose blue or green.

-

	blue	green	red	yellow	same color as friend
you	4	3	2	×	0
Barbara	×	×	4	3	0

- But then, your unique optimal choice is blue.

- So, under common belief in rationality, you can only rationally wear blue.

New Scenario

- Barbara has same preferences over colors as you.
- Barbara likes to wear the same color as you, whereas you dislike this.
-

	blue	green	red	yellow	same color as friend
you	4	3	2	1	0
Barbara	4	3	2	1	5

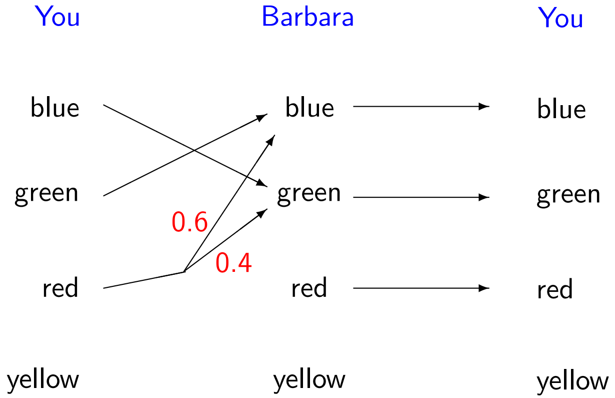
- Which color(s) can you rationally choose under common belief in rationality?

	blue	green	red	yellow	same color as friend
you	4	3	2	1	0
Barbara	4	3	2	1	5

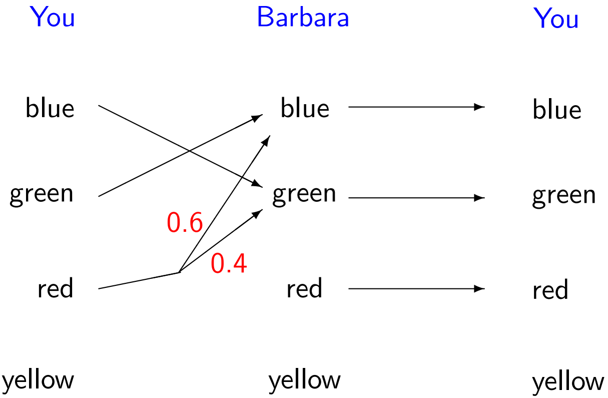
- If you choose rationally, you will not choose yellow.
- If you believe that Barbara chooses rationally, and believe that Barbara believes that you choose rationally, then you believe that Barbara will not choose yellow either.
-

	blue	green	red	yellow	same color as friend
you	4	3	2	×	0
Barbara	4	3	2	×	5

Beliefs diagram



	blue	green	red	yellow	same color as friend
you	4	3	2	×	0
Barbara	4	3	2	×	5



- The belief hierarchy that starts at your choice blue expresses common belief in rationality.
- Similarly, the belief hierarchies that start at your choices green and red also express common belief in rationality.
- So, you can rationally choose blue, green and red under common belief in rationality.

Epistemic model

- Writing down a belief hierarchy explicitly is impossible. You must write down
- your belief about the opponents' choices
- your belief about what your opponents believe about their opponents' choices,
- a belief about what the opponents believe that their opponents believe about the other players' choices,
- and so on, ad infinitum.
- Is there an easy way to encode a belief hierarchy?

- A belief hierarchy for you consists of a first-order belief, a second-order belief, a third-order belief, and so on.

- In a belief hierarchy, you hold a belief about
- the opponents' choices,
- the opponents' first-order beliefs,
- the opponents' second-order beliefs,
- and so on.

- Hence, in a belief hierarchy you hold a belief about
- the opponents' choices, and the opponents' belief hierarchies.

- Following Harsanyi (1967–1968), call a belief hierarchy a type.
- Then, a type holds a belief about the opponents' choices and the opponents' types.

- Let $I = \{1, ..., n\}$ be the set of players.
- For every player i, let C_i be the finite set of choices.

Definition (Epistemic model)

A finite epistemic model specifies for every player i a finite set T_i of possible types.

Moreover, for every type t_i it specifies a probabilistic belief $b_i(t_i)$ over the set $C_{-i} \times T_{-i}$ of opponents' choice-type combinations.

- Implicit epistemic model: For every type, we can derive the belief hierarchy induced by it.
- This is the model as used by Tan and Werlang (1988).
- Builds upon work by Harsanyi (1967–1968), Armbruster and Böge (1979), Böge and Eisele (1979), and Bernheim (1984).

Common Belief in Rationality

Formal definition

- Remember: A type t_i holds a probabilistic belief $b_i(t_i)$ over the set $C_{-i} \times T_{-i}$ of opponents' choice-type combinations.
- For a choice c_i, let

$$u_i(c_i, t_i) := \sum_{(c_{-i}, t_{-i}) \in C_{-i} \times T_{-i}} b_i(t_i)(c_{-i}, t_{-i}) \cdot u_i(c_i, c_{-i})$$

  be the expected utility that type t_i obtains by choosing c_i.
- Choice c_i is optimal for type t_i if

$$u_i(c_i, t_i) \geq u_i(c_i', t_i) \text{ for all } c_i' \in C_i.$$

Definition (Belief in the opponents' rationality)

Type t_i believes in the opponents' rationality if his belief $b_i(t_i)$ only assigns positive probability to opponents' choice-type pairs (c_j, t_j) where choice c_j is optimal for type t_j.

Definition (Common belief in rationality)

(Induction start) Type t_i expresses 1-fold belief in rationality if t_i believes in the opponents' rationality.

(Inductive step) For every $k \geq 2$, type t_i expresses k-fold belief in rationality if t_i only assigns positive probability to opponents' types that express $(k-1)$-fold belief in rationality.

Type t_i expresses common belief in rationality if t_i expresses k-fold belief in rationality for all k.

- Based on Tan and Werlang (1988) .

Recursive Procedure

- Suppose we wish to find those choices you can rationally make under common belief in rationality.
- Is there a recursive procedure that helps us find these choices?
- Based on following result:

Lemma (Pearce (1984))

A choice c_i is optimal for some probabilistic belief about the opponents' choices, if and only if, c_i is not strictly dominated by any randomized choice.

- Here, a randomized choice r_i for player i is a probability distribution on i's choices.
- Choice c_i is strictly dominated by the randomized choice r_i if

$$u_i(c_i, c_{-i}) < u_i(r_i, c_{-i})$$

for every opponents' choice-combination $c_{-i} \in C_{-i}$.

Definition (Iterated elimination of strictly dominated choices)

Consider a finite static game Γ.

(Induction start) Let $\Gamma^0 := \Gamma$ be the original game.

(Inductive step) For every $k \geq 1$, let Γ^k be the game which results if we eliminate from Γ^{k-1} all choices that are strictly dominated within Γ^{k-1}.

- This procedure terminates within finitely many steps. That is, there is some K with $\Gamma^{K+1} = \Gamma^K$.
- The choices in Γ^K are said to survive iterated elimination of strictly dominated choices.
- It always yields a nonempty set of choices for all players.
- The final output does not depend on the order by which we eliminate choices.

Definition (Iterated elimination of strictly dominated choices)

Consider a finite static game Γ.

(Induction start) Let $\Gamma^0 := \Gamma$ be the original game.

(Inductive step) For every $k \geq 1$, let Γ^k be the game which results if we eliminate from Γ^{k-1} all choices that are strictly dominated within Γ^{k-1}.

- In two-player games, it yields exactly the rationalizable choices, as defined by Bernheim (1984) and Pearce (1984).
- For games with more than two players, rationalizability requires player i's belief about player j's choice to be stochastically independent from his belief about player k's choice.
- The procedure does not impose this independence condition.
- For games with more than two players, this procedure yields correlated rationalizability (Brandenburger and Dekel (1987)).

Theorem (Tan and Werlang (1988))

(1) For every $k \geq 1$, the choices that are optimal for a type that expresses up to k-fold belief in rationality are exactly those choices that survive $(k+1)$-fold elimination of strictly dominated choices.

(2) The choices that are optimal for a type that expresses common belief in rationality are exactly those choices that survive iterated elimination of strictly dominated choices.

Corollary (Common belief in rationality is always possible)

We can always construct an epistemic model in which all types express common belief in rationality.

Nash equilibrium

- Nash equilibrium has dominated game theory for many years.
- But until the rise of Epistemic Game Theory it remained unclear what Nash equilibrium assumes about the reasoning of the players.
- We will now investigate Nash equilibrium from an epistemic point of view.
- We will see that Nash equilibrium requires more than just common belief in rationality.
- We show that Nash equilibrium can be epistemically characterized by

  common belief in rationality + simple belief hierarchy.

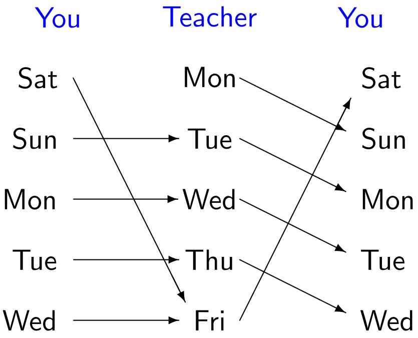
- However, the condition of a simple belief hierarchy is quite unnatural, and overly restrictive.

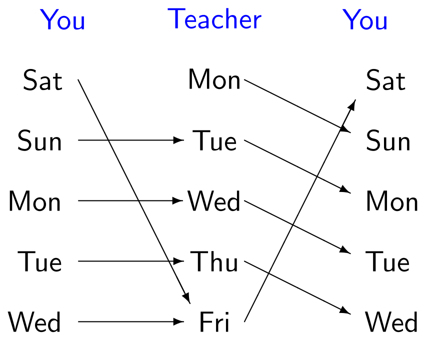
Example: Teaching a lesson

Story

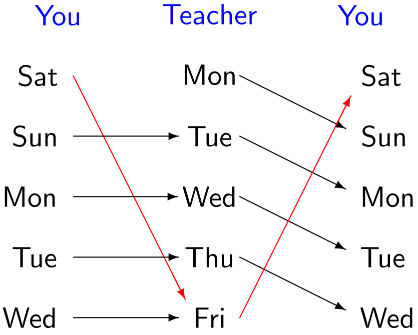
- It is Friday, and your biology teacher tells you that he will give you a surprise exam next week.
- You must decide on what day you will start preparing for the exam.
- In order to pass the exam, you must study for at least two days.
- To write the perfect exam, you must study for at least six days. In that case, you will get a compliment by your father.
- Passing the exam increases your utility by 5.
- Failing the exam increases the teacher's utility by 5.
- Every day you study decreases your utility by 1, but increases the teacher's utility by 1.
- A compliment by your father increases your utility by 4.

		Teacher				
		Mon	Tue	Wed	Thu	Fri
You	Sat	3, 2	2, 3	1, 4	0, 5	3, 6
	Sun	−1, 6	3, 2	2, 3	1, 4	0, 5
	Mon	0, 5	−1, 6	3, 2	2, 3	1, 4
	Tue	0, 5	0, 5	−1, 6	3, 2	2, 3
	Wed	0, 5	0, 5	0, 5	−1, 6	3, 2

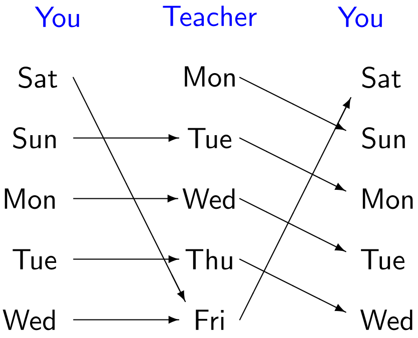




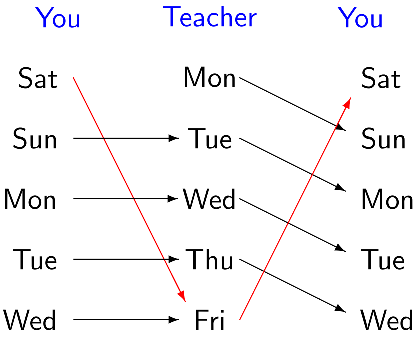
- Under common belief in rationality, you can rationally choose any day to start studying.
- Yet, some choices are supported by a simple belief hierarchy, whereas other choices are not.



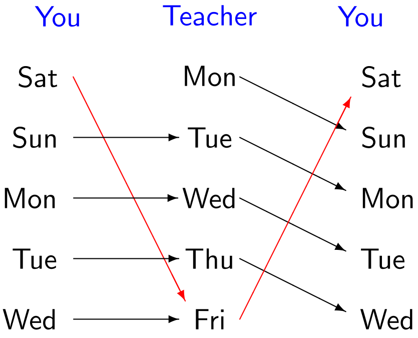
- Consider the belief hierarchy that supports your choices Saturday and Wednesday.
- This belief hierarchy is entirely generated by the belief σ_2 that the teacher puts the exam on Friday, and the belief σ_1 that you start studying on Saturday.
- We call such a belief hierarchy simple.
- In fact, $(\sigma_1, \sigma_2) =$ (Sat, Fri) is a Nash equilibrium.



- The belief hierarchies that support your choices Sunday, Monday and Tuesday are certainly not simple. Consider, for instance, the belief hierarchy that supports your choice Sunday. There,
- you believe that the teacher puts the exam on Tuesday,
- but you believe that the teacher believes that you believe that the teacher will put the exam on Wednesday.
- Hence, this belief hierarchy cannot be generated by a single belief σ_2 about the teacher's choice.



- One can show: Your choices Sunday, Monday and Tuesday cannot be supported by simple belief hierarchies that express common belief in rationality.
- Your choices Sunday, Monday and Tuesday cannot be optimal in any Nash equilibrium of the game.



Summarizing

- Your choices Saturday and Wednesday are the only choices that are optimal for a simple belief hierarchy that expresses common belief in rationality.
- These are also the only choices that are optimal for you in any Nash equilibrium of the game.

Simple belief hierarchies

- A belief hierarchy is called simple if it is generated by a single combination of beliefs $\sigma_1, ..., \sigma_n$.

Definition (Belief hierarchy generated by $(\sigma_1, ..., \sigma_n)$)

For every player i, let σ_i be a probabilistic belief about i's choice.

The belief hierarchy for player i that is generated by $(\sigma_1, ..., \sigma_n)$ states that

(1) player i has belief σ_j about player j's choice,

(2) player i believes that player j has belief σ_k about player k's choice,

(3) player i believes that player j believes that player k has belief σ_l about player l's choice,

and so on.

Definition (Simple belief hierarchy)

Consider an epistemic model, and a type t_i within it.

Type t_i has a simple belief hierarchy, if its belief hierarchy is generated by some combination of beliefs $(\sigma_1, ..., \sigma_n)$.

- A player i with a simple belief hierarchy has the following properties:
- He believes that every opponent is correct about his belief hierarchy.
- He believes that every opponent j has the same belief about player k as he has.
- His belief about j's choice is stochastically independent from his belief about k's choice.

Nash equilibrium

- Nash (1950, 1951) phrased his equilibrium notion in terms of randomized choices (or, mixed strategies) $\sigma_1, ..., \sigma_n$, where $\sigma_i \in \Delta(C_i)$ for every player i.
- Following Aumann and Brandenburger (1995), we interpret $\sigma_1, ..., \sigma_n$ as beliefs.

Definition (Nash equilibrium)

A combination of beliefs $(\sigma_1, ..., \sigma_n)$, where $\sigma_i \in \Delta(C_i)$ for every player i, is a Nash equilibrium if for every player i, the belief σ_i only assigns positive probability to choices c_i that are optimal under the belief $\sigma_{-i} \in \Delta(C_{-i})$.

- Here, $\sigma_{-i} \in \Delta(C_{-i})$ is the probability distribution given by

$$\sigma_{-i}(c_{-i}) := \prod_{j \neq i} \sigma_j(c_j)$$

for every $c_{-i} = (c_j)_{j \neq i}$ in C_{-i}.

Theorem (Characterization of Nash equilibrium)

Consider a type t_i with a simple belief hierarchy, generated by the combination $(\sigma_1, ..., \sigma_n)$ of beliefs.

Then, type t_i expresses common belief in rationality, if and only if, the combination of beliefs $(\sigma_1, ..., \sigma_n)$ is a Nash equilibrium.

- Other epistemic foundations of Nash equilibrium can be found in Spohn (1982), Brandenburger and Dekel (1987, 1989), Tan and Werlang (1988), Aumann and Brandenburger (1995), Polak (1999), Asheim (2006), Perea (2007), Barelli (2009) and Bach and Tsakas (2014).
- All these foundations involve some correct beliefs assumption: You believe that your opponents are correct about your first-order belief.
- Not all layers of common belief in rationality are needed to obtain Nash equilibrium.

How reasonable is Nash equilibrium?

- We have seen that a Nash equilibrium makes the following assumptions:
- you believe that your opponents are correct about the beliefs that you hold;
- you believe that player j holds the same belief about player k as you do;
- your belief about player j's choice is independent from your belief about player k's choice.
- Each of these conditions is actually very questionable.
- Therefore, Nash equilibrium is perhaps not such a natural concept after all.

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